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INVARIANT THEORY OF HOMOGENEOUS CONTACT TRANSFORMATIONS

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Introduction. Contact transformations have had their applications primarily in dynamics and to a rather small extent in geometry. This is no doubt due to the fact that the geometrical object to which a contact transformation may be applied is of a more complicated nature than the objects to which we apply extended point transformations. The present paper, although far from completely solving the problem, indicates how the invariant theory of homogeneous contact transformations may be formulated.* We first define a contact frame (§4) which is sufficient for the study of Grassmann's forms of exterior multiplication. We then introduce a linear connection (§§5, 6, 7) for the affine invariant theory and finally we introduce a metric, or rather a part of it, that is necessary for the study of the metric properties of curves. Incidentally we generalize the Poisson brackets of both kinds and find a new and rather simple way of characterizing homogeneous contact transformations.

1. **Homogeneous contact transformations.** A homogeneous contact transformation is defined by the equations

$$\bar{x}^i = \varphi^i(x^1, \cdots, x^n; p_1 \cdots p_n) \equiv \varphi^i(x, p), \qquad \bar{p}_i = \psi_i(x; p) \tag{1.1}$$

where φ^i and ψ_i are such that the condition[1]

$$\bar{p}_i d\bar{x}^i = p_\alpha dx^\alpha, \tag{1.2}$$

that is

$$\psi_i\left(\frac{\partial \varphi^i}{\partial x^\alpha}\, dx^\alpha + \frac{\partial \varphi^i}{\partial p_\alpha}\, dp_\alpha\right) = p_\alpha dx^\alpha \tag{1.3}$$

holds for arbitrary values of the differentials dx^α, dp_α. From (1.1) we have

$$d\bar{x}^i = \frac{\partial \varphi^i}{\partial x^\alpha}\, dx^\alpha + \frac{\partial \varphi^i}{\partial p_\alpha}\, dp_\alpha, \qquad d\bar{p}_i = \frac{\partial \psi_i}{\partial x^\alpha}\, dx^\alpha + \frac{\partial \psi_i}{\partial p_\alpha}\, dp_\alpha \tag{1.4}$$

and consequently the above requirement leads to the conditions

$$\bar{p}_i \frac{\partial \bar{x}^i}{\partial x^\alpha} = p_\alpha, \qquad \bar{p}_i \frac{\partial \bar{x}^i}{\partial p_\alpha} = 0. \tag{1.5}$$

* We are aware of only one attempt in this direction, due to T. Hosokawa (Journal of Hokkaido Imperial Univ., Series 1, V. II, No. 3) but he attacks the problem under very severe restrictions and makes assumptions that are not necessarily justifiable.

[1] The customary summation convention is used throughout this paper.

It can be shown that φ^i and ψ_i are homogeneous in the p's of degrees zero and one respectively.[2]

Differentiating each of the equations (1.5) with respect to x^β and p_β, we obtain from the resulting equations the equations

$$\frac{\partial \bar{p}_i}{\partial x^\beta}\frac{\partial \bar{x}^i}{\partial x^\alpha} - \frac{\partial \bar{p}_i}{\partial x^\alpha}\frac{\partial \bar{x}^i}{\partial x^\beta} = 0, \qquad \frac{\partial \bar{p}_i}{\partial p_\beta}\frac{\partial \bar{x}^i}{\partial x^\alpha} - \frac{\partial \bar{p}_i}{\partial x^\alpha}\frac{\partial \bar{x}^i}{\partial p_\beta} = \delta^\beta_\alpha,$$

(1.6)

$$\frac{\partial \bar{p}_i}{\partial p_\beta}\frac{\partial \bar{x}^i}{\partial p_\alpha} - \frac{\partial \bar{p}_i}{\partial p_\alpha}\frac{\partial \bar{x}^i}{\partial p_\beta} = 0.$$

From (1.4) and (1.6) we have

$$d\bar{x}^i \frac{\partial \bar{p}_i}{\partial p_\alpha} - d\bar{p}_i \frac{\partial \bar{x}^i}{\partial p_\alpha} = dx^\alpha, \qquad d\bar{x}^i \frac{\partial \bar{p}_i}{\partial x^\alpha} - d\bar{p}_i \frac{\partial \bar{x}^i}{\partial x^\alpha} = -dp_\alpha. \tag{1.7}$$

Since each contact transformation has a unique inverse, we have

$$dx^\alpha = \frac{\partial x^\alpha}{\partial \bar{x}^i}\, d\bar{x}^i + \frac{\partial x^\alpha}{\partial \bar{p}_i}\, d\bar{p}_i, \qquad dp_\alpha = \frac{\partial p_\alpha}{\partial \bar{x}^i}\, d\bar{x}^i + \frac{\partial p_\alpha}{\partial \bar{p}_i}\, d\bar{p}_i.$$

From these equations and (1.7), since the differentials $d\bar{x}^i$, $d\bar{p}_i$ are independent, we have

$$\frac{\partial \bar{p}_i}{\partial p_\alpha} = \frac{\partial x^\alpha}{\partial \bar{x}^i}, \qquad \frac{\partial p_\alpha}{\partial \bar{p}_i} = \frac{\partial \bar{x}^i}{\partial x^\alpha}, \qquad \frac{\partial x^\alpha}{\partial \bar{p}_i} + \frac{\partial \bar{x}^i}{\partial p_\alpha} = 0, \qquad \frac{\partial \bar{p}_i}{\partial x^\alpha} + \frac{\partial p_\alpha}{\partial \bar{x}^i} = 0. \tag{1.8}$$

From these equations and (1.6) we obtain

$$(\overline{p_\alpha, p_\beta}) \equiv \frac{\partial p_\alpha}{\partial \bar{p}_i}\frac{\partial p_\beta}{\partial \bar{x}^i} - \frac{\partial p_\beta}{\partial \bar{p}_i}\frac{\partial p_\alpha}{\partial \bar{x}^i} = 0, \qquad (\overline{p_\alpha, x^\beta}) \equiv \frac{\partial p_\alpha}{\partial \bar{p}_i}\frac{\partial x^\beta}{\partial \bar{x}^i} - \frac{\partial x^\beta}{\partial \bar{p}_i}\frac{\partial p_\alpha}{\partial \bar{x}^i} = \delta^\beta_\alpha,$$

(1.9)

$$(\overline{x^\alpha, x^\beta}) \equiv \frac{\partial x^\alpha}{\partial \bar{p}_i}\frac{\partial x^\beta}{\partial \bar{x}^i} - \frac{\partial x^\beta}{\partial \bar{p}_i}\frac{\partial x^\alpha}{\partial \bar{x}^i} = 0.$$

These are equations for the inverse of (1.1). If we put

$$(u, v) \equiv \frac{\partial u}{\partial p_\alpha}\frac{\partial v}{\partial x^\alpha} - \frac{\partial u}{\partial p_\alpha}\frac{\partial u}{\partial x^\alpha}, \tag{1.10}$$

then for the transformation (1.1) we have

$$(\bar{x}^i, \bar{x}^j) = 0, \qquad (\bar{p}_i, \bar{x}^j) = \delta^j_i, \qquad (\bar{p}_i, \bar{p}_j) = 0. \tag{1.11}$$

Equations (1.8), or their equivalent (1.11), constitute a necessary and sufficient condition that equations (1.1) define a homogeneous contact transformation.[3]

2. Configurations and their transformations. Proceeding from equations (1.4), we consider $2n$ functions Λ^α, $\Lambda_\alpha (\alpha = 1, \cdots n)$ of the x's and p's, and

[2] Cf. *C. G.*, p. 240. A reference of this kind is to Eisenhart's *Continuous Groups of Transformations.*

[3] *C. G.*, p. 250.

under a contact transformation (1.1) the $2n$ functions $\bar{\Lambda}^i$, $\bar{\Lambda}_i$ of the $\bar{x}$'s and $\bar{p}$'s defined by

$$\bar{\Lambda}^i = \frac{\partial \bar{x}^i}{\partial x^\alpha}\Lambda^\alpha + \frac{\partial \bar{x}^i}{\partial p_\alpha}\Lambda_\alpha, \qquad \bar{\Lambda}_i = \frac{\partial \bar{p}_i}{\partial x^\alpha}\Lambda^\alpha + \frac{\partial \bar{p}_i}{\partial p_\alpha}\Lambda_\alpha. \tag{2.1}$$

We say that the Λ's are the components of an entity in V_n, a *configuration of the first order*, which is transformed into the entity of components $\bar{\Lambda}$ under the contact transformation. Thus the differentials dx^α, dp_α are transformed into $d\bar{x}^i$, $d\bar{p}_i$, and are the components of a configuration. Also in consequence of (1.5) the configuration

$$\Lambda^\alpha = 0 \qquad \Lambda_\alpha = p_\alpha \tag{2.2}$$

is transformed into $\bar{\Lambda}^i = 0$, $\bar{\Lambda}_i = \bar{p}_i$. Likewise for any configuration we have

$$\bar{p}_i\bar{\Lambda}^i = p_\alpha\Lambda^\alpha. \tag{2.3}$$

In consequence of (1.6), and analogously to (1.7), equations (2.1) may be written in the form

$$\bar{\Lambda}^i\frac{\partial \bar{p}^i}{\partial p_\alpha} - \bar{\Lambda}_i\frac{\partial \bar{x}^i}{\partial p_\alpha} = \Lambda^\alpha, \qquad \bar{\Lambda}^i\frac{\partial \bar{p}_i}{\partial x^\alpha} - \bar{\Lambda}_i\frac{\partial \bar{x}^i}{\partial x^\alpha} = -\Lambda_\alpha. \tag{2.4}$$

If we differentiate the *scalar* relation

$$\bar{f}(\bar{x}, \bar{p}) = f(x, p)$$

with respect to x^α and p_α, and compare the resulting equations with (2.4), we see that the configuration

$$\Lambda^\alpha = -\frac{\partial f}{\partial p_\alpha} \qquad \Lambda_\alpha = \frac{\partial f}{\partial x^\alpha} \tag{2.5}$$

is transformed into

$$\bar{\Lambda}^i = -\frac{\partial \bar{f}}{\partial \bar{p}_i}, \qquad \bar{\Lambda}_i = \frac{\partial \bar{f}}{\partial \bar{x}^i}.$$

If Λ^α, Λ_α and M^α, M_α are the components of two configurations in the x's and p's, we have from (2.1) and (1.6) that

$$\bar{\Lambda}^i\bar{\mathrm{M}}_i - \bar{\Lambda}_i\bar{\mathrm{M}}^i = \Lambda^\alpha\mathrm{M}_\alpha - \Lambda_\alpha\mathrm{M}^\alpha. \tag{2.6}$$

When in particular, we take for Λ and M expressions of the form (2.5) in terms of two functions f and g, we have

$$(f, g) = (\bar{f}, \bar{g})$$

which is the well-known result that the parenthesis of Poisson of two functions is invariant under a homogeneous contact transformation.[4] Also (2.3) is a special case of (2.6) for Λ^α, Λ_α and 0, p_α.

[4] *C. G.*, p. 250.

If we put

$$A^{\alpha\beta} = \Lambda^{\alpha}\mathrm{M}^{\beta}, \qquad A^{\alpha}{}_{\beta} = \Lambda^{\alpha}\mathrm{M}_{\beta}, \qquad A_{\beta}{}^{\alpha} = \Lambda_{\beta}\mathrm{M}^{\alpha}, \qquad A_{\alpha\beta} = \Lambda_{\alpha}\mathrm{M}_{\beta} \tag{2.7}$$

and similarly for $\bar{\Lambda}$ and $\overline{\mathrm{M}}$, we have by means of (2.1)

$$\begin{aligned}
\bar{A}^{ij} &= \frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial \bar{x}^j}{\partial x^\beta}A^{\alpha\beta} + \frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial \bar{x}^j}{\partial p_\beta}A^{\alpha}{}_{\beta} + \frac{\partial \bar{x}^i}{\partial p_\alpha}\frac{\partial \bar{x}^j}{\partial x^\beta}A_{\alpha}{}^{\beta} + \frac{\partial \bar{x}^i}{\partial p_\alpha}\frac{\partial \bar{x}^j}{\partial p_\beta}A_{\alpha\beta} \\
\bar{A}^{i}{}_{j} &= \frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial \bar{p}_j}{\partial x^\beta}A^{\alpha\beta} + \frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial \bar{p}_j}{\partial p_\beta}A^{\alpha}{}_{\beta} + \frac{\partial \bar{x}^i}{\partial p_\alpha}\frac{\partial \bar{p}_j}{\partial x^\beta}A_{\alpha}{}^{\beta} + \frac{\partial \bar{x}^i}{\partial p_\alpha}\frac{\partial \bar{p}_j}{\partial p_\beta}A_{\alpha\beta} \\
\bar{A}_{i}{}^{j} &= \frac{\partial \bar{p}_i}{\partial x^\alpha}\frac{\partial \bar{x}^j}{\partial x^\beta}A^{\alpha\beta} + \frac{\partial \bar{p}_i}{\partial x^\alpha}\frac{\partial \bar{x}^j}{\partial p_\beta}A^{\alpha}{}_{\beta} + \frac{\partial \bar{p}_i}{\partial p_\alpha}\frac{\partial \bar{x}^j}{\partial x^\beta}A_{\alpha}{}^{\beta} + \frac{\partial \bar{p}_i}{\partial p_\alpha}\frac{\partial \bar{x}^j}{\partial p_\beta}A_{\alpha\beta} \\
\bar{A}_{ij} &= \frac{\partial \bar{p}_i}{\partial x^\alpha}\frac{\partial \bar{p}_j}{\partial x^\beta}A^{\alpha\beta} + \frac{\partial \bar{p}_i}{\partial x^\alpha}\frac{\partial \bar{p}_j}{\partial p_\beta}A^{\alpha}{}_{\beta} + \frac{\partial \bar{p}_i}{\partial p_\alpha}\frac{\partial \bar{p}_j}{\partial x^\beta}A_{\alpha}{}^{\beta} + \frac{\partial \bar{p}_i}{\partial p_\alpha}\frac{\partial \bar{p}_j}{\partial p_\beta}A_{\alpha\beta}.
\end{aligned} \tag{2.8}$$

Any sets of quantities satisfying these equations under a contact transformation we call the components of a *configuration of the second order*. Later (§3) we give the definition of configurations of higher order.

From (2.8) we have for the particular case

$$A^{\alpha\beta} = 0, \qquad A^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta}, \qquad A_{\alpha}{}^{\beta} = -\delta_{\alpha}{}^{\beta}, \qquad A_{\alpha\beta} = 0 \tag{2.9}$$

the components $\bar{A}$ are of the same form. Thus (2.9) define a numerical configuration of the second order.

From (2.1) and (2.8) it follows that

$$\Lambda^{\alpha}A_{\alpha}{}^{\beta} - \Lambda_{\alpha}A^{\alpha\beta}, \ \Lambda^{\alpha}A_{\alpha\beta} - \Lambda_{\alpha}A^{\alpha}{}_{\beta} \tag{2.10}$$

are the components of a configuration of the first, order, as are also

$$A^{\alpha}{}_{\beta}\Lambda^{\beta} - A^{\alpha\beta}\Lambda_{\beta}, \ A_{\alpha\beta}\Lambda^{\beta} - A_{\alpha}{}^{\beta}\Lambda_{\beta}. \tag{2.11}$$

In particular for the case (2.2) we have that

$$p_\alpha A^{\alpha\beta}, \ p_\alpha A^{\alpha}{}_{\beta} \text{ and } A^{\alpha\beta}p_\beta, \ A_{\alpha}{}^{\beta}p_\beta \tag{2.12}$$

are components of two configurations of the first order.

3. **A subgroup of the general analytic group.** The group of homogeneous contact transformations may be looked upon as a sub-group of the general analytic group in $2n$ variables. Let $x^1, \cdots, x^{2n}$ be these variables with the understanding that $x^{n+\alpha} = p_\alpha$. We denote the configuration Λ^α, Λ_α by Λ^A, with the understanding that a superscript $n + i$ is replaced by a subscript i. In this notation equations (2.1) become

$$\bar{\Lambda}^A = \Lambda^B \frac{\partial \bar{x}^A}{\partial x^B} \qquad A, B = 1, \cdots, 2n \tag{3.1}$$

and (2.8) are

$$\bar{A}^{AB} = A^{CD}\frac{\partial \bar{x}^A}{\partial x^C}\frac{\partial \bar{x}^B}{\partial x^D}. \tag{3.2}$$

By the use of this notation we may write down readily the equations of transformation of a configuration of any order.

The configuration (2.9) may also be written as

$$\epsilon^{AB} = +1 \text{ for } B = A + n, \quad -1 \text{ for } B = A - n, \quad 0 \text{ otherwise.} \tag{3.3}$$

If we require that the configuration ϵ^{AB} be invariant under the transformation (3.2) we obtain (1.11). Hence:

[3.1] *A necessary and sufficient condition that an analytic transformation* $\bar{x}^A = \bar{x}^A(x)$ *be a homogeneous contact transformation is that the antisymmetric configuration* ϵ^{AB} *be invariant under the transformation.*

Since $|\epsilon^{AB}| = +1$, it follows from (3.2) for this configuration that $\Delta^2 = 1$, Δ being the Jacobian of the transformation. For an infinitesimal transformation $\bar{x}^A = \bar{x}^A + \xi^A\epsilon$, $\Delta = 1 + A_0\epsilon^2$ and since the transformation is continuous we have:

[3.2] *The Jacobian of a homogeneous contact transformation is equal to* $+1$.

We refer to the foregoing configurations as contravariant, and to those, as A_{AB}, as covariant, understanding that for $A = n + \alpha$ the index is a superscript. Its law of transformation is

$$\bar{A}_{AB} = A_{CD}\frac{\partial x^C}{\partial \bar{x}^A}\frac{\partial x^D}{\partial x^B}. \tag{3.4}$$

In particular we have the skew-symmetric numerical configuration ϵ_{AB} taking the values as in (3.3). If we put $\Lambda'_A = \epsilon_{AB}\Lambda^B$, we have $\Lambda'_\alpha = \Lambda_\alpha$, $\Lambda'^\alpha = -\Lambda^\alpha$. Thus equations (2.4) may be written, in consequence of (1.8)

$$\Lambda'_A = \Lambda'_B\frac{\partial x^B}{\partial \bar{x}^A}. \tag{3.5}$$

Likewise we have mixed configurations, or $A^A{}_B$, the first index being a subscript i for $A = n + i$, etc.

The relations

$$\epsilon^{AC}\epsilon_{BC} = -\epsilon^{AC}\epsilon_{CB} = \epsilon^{CA}\epsilon_{CB} = \delta^A_B \tag{3.6}$$

will be useful in what follows.

An important subgroup of the full linear group is what Weyl calls the *complex* group; it is of determinant $+1$ and bears the same relation to the non-singular skew-symmetric bilinear form $\{A, B\} = A^\alpha B_\alpha - A_\alpha B^\alpha$ that the orthogonal group does to the symmetric form $(A, B) = A^\alpha B_\alpha + A_\alpha B^\alpha$. Therefore the basic scalar invariant of the group of homogeneous contact transformations is $dx^\alpha \delta p_\alpha - dp_\alpha \delta x^\alpha$; it completely characterises the group of homogeneous contact transformations.

4. **The contact frame.** In order to determine the significance of the components Λ^α, Λ_α of a configuration in a V_n of coördinates x^α we consider the

special contact transformations, which are extended point transformations, where φ^i in (1.1) do not involve the p's; we write them thus

$$x'^i = \varphi^i(x), \qquad p'_i = \frac{\partial x^\beta}{\partial x'^i} p_\beta, \tag{4.1}$$

the latter being a consequence of (1.5). In this case equations (2.1) become

$$\Lambda'^i = \frac{\partial x'^i}{\partial x^\alpha} \Lambda^\alpha \qquad \Lambda'_i = \frac{\partial p'_i}{\partial x^\alpha} \Lambda^\alpha + \frac{\partial p'_i}{\partial p_\alpha} \Lambda_\alpha = \frac{\partial p'_i}{\partial x^\alpha} \Lambda^\alpha + \frac{\partial x^\alpha}{\partial x'^i} \Lambda_\alpha, \tag{4.2}$$

the latter following from (1.8). From the first of these equations it follows that Λ^α, which may involve the p's, are components of a contravariant vector, and from the second of (4.1) that p_α are the components of a covariant vector.

Suppose an affine connection is assigned to the V_n by means of the equations

$$\frac{\partial^2 x^\beta}{\partial x'^i \partial x'^j} = \Gamma'^{\,k}_{i\,j} \frac{\partial x^\beta}{\partial x'^k} - \Gamma^\beta_{\gamma\delta} \frac{\partial x^\gamma}{\partial x'^i} \frac{\partial x^\delta}{\partial x'^j}. \tag{4.3}$$

From these equations and (4.1) we obtain

$$\frac{\partial p'_i}{\partial x^\alpha} = \frac{\partial x'^j}{\partial x^\alpha} p_\beta \left(\Gamma'^{\,k}_{i\,j} \frac{\partial x^\beta}{\partial x'^k} - \Gamma^\beta_{\gamma\delta} \frac{\partial x^\gamma}{\partial x'^i} \frac{\partial x^\delta}{\partial x'^j} \right) = \frac{\partial x'^j}{\partial x^\alpha} \Gamma'^{\,k}_{i\,j} p'_k - \frac{\partial x^\gamma}{\partial x'^i} \Gamma^\beta_{\gamma\alpha} p_\beta. \tag{4.4}$$

By means of this result the second set of equations (4.1) may be written

$$\Lambda'_i - \Lambda'^j \Gamma'^{\,k}_{i\,j} p'_k = (\Lambda_\alpha - \Lambda^\beta \Gamma^\gamma_{\alpha\beta} p_\gamma) \frac{\partial x^\alpha}{\partial x'^i}.$$

Consequently if we put

$$\lambda_\alpha = \Lambda_\alpha - \Lambda^\beta \Gamma^\gamma_{\alpha\beta} p_\gamma, \tag{4.5}$$

then λ_α are the components of a covariant vector in V_n. Accordingly if we have a configuration Λ^α, Λ_α, then $\lambda^\alpha = \Lambda^\alpha$ and λ^α given by (4.5) are the components of two vectors in V_n, and conversely if we have two vectors we get a configuration.

The preceding result followed the introduction of an affine connection in V_n and the use of extended point transformations. In any case a configuration Λ^A may be assigned a set of components λ^A defined by the equations

$$\lambda^\alpha = \Lambda^\alpha, \qquad \lambda_\alpha = \Lambda_\alpha - \Gamma_{\alpha\beta} \Lambda^\beta \tag{4.6}$$

where $\Gamma_{\alpha\beta}$ are functions of the x's and p's homogeneous of degree one in the p's and symmetric α and β. We require that this definition be invariant not only under extended point transformations but under the full group of homogeneous contact transformations. Hence from equations (2.1) we must have

$$\begin{aligned} \bar\lambda^i &= \lambda^\alpha \left(\frac{\partial \bar x^i}{\partial x^\alpha} + \Gamma_{\beta\alpha} \frac{\partial \bar x^i}{\partial p_\beta} \right) + \lambda_\alpha \frac{\partial \bar x^i}{\partial p_\alpha} \\ \bar\lambda_j + \bar\Gamma_{ji} \bar\lambda^i &= \lambda^\alpha \left(\frac{\partial \bar p_j}{\partial x^\alpha} + \Gamma_{\beta\alpha} \frac{\partial \bar p_j}{\partial p_\beta} \right) + \lambda_\alpha \frac{\partial \bar p_j}{\partial p_\alpha}. \end{aligned} \tag{4.7}$$

From these equations we have in consequence of (1.8)

$$\bar{\lambda}_j - \lambda_\alpha\left(\frac{\partial x^\alpha}{\partial \bar{x}^j} + \bar{\Gamma}_{ji}\frac{\partial x^\alpha}{\partial \bar{p}_i}\right) + \lambda^\alpha\left[\bar{\Gamma}_{ji}\left(\frac{\partial \bar{x}^i}{\partial x^\alpha} + \Gamma_{\beta\alpha}\frac{\partial \bar{x}^i}{\partial p_\beta}\right) - \left(\frac{\partial \bar{p}_j}{\partial x^\alpha} + \Gamma_{\beta\alpha}\frac{\partial \bar{p}_j}{\partial p^\beta}\right)\right] = 0.$$

In order that under an extended point transformation λ_α be transformed into $\bar{\lambda}_j$, we must have (changing indices)

$$\bar{\lambda}_i = \lambda_\alpha\left(\frac{\partial x^\alpha}{\partial \bar{x}^i} + \bar{\Gamma}_{ij}\frac{\partial x^\alpha}{\partial \bar{p}_j}\right) \tag{4.8}$$

and

$$\bar{\Gamma}_{ij}\left(\frac{\partial \bar{x}^j}{\partial x^\alpha} + \Gamma_{\alpha\beta}\frac{\partial \bar{x}^j}{\partial p_\beta}\right) = \frac{\partial \bar{p}_i}{\partial x^\alpha} + \Gamma_{\alpha\beta}\frac{\partial \bar{p}_i}{\partial p_\beta}. \tag{4.9}$$

The functions $\Gamma_{\alpha\beta}(x, p)$ are the components of an entity we call a *contact frame* and (4.9) is its law of transformation. We must show that the components of the contact frame are uniquely determined under any contact transformation and that the symmetry property is preserved. By means of (1.8), (4.9) may be written as

$$\Gamma_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial \bar{x}^i} + \bar{\Gamma}_{ij}\frac{\partial x^\alpha}{\partial \bar{p}_j}\right) = \frac{\partial p_\beta}{\partial \bar{x}^i} + \bar{\Gamma}_{ij}\frac{\partial p_\beta}{\partial \bar{p}_j} \tag{4.10}$$

for which and (4.9) we get

$$\left(\frac{\partial x^\alpha}{\partial \bar{x}^j} + \bar{\Gamma}_{jh}\frac{\partial x^\alpha}{\partial \bar{p}_h}\right)\left(\frac{\partial \bar{x}^i}{\partial x^\alpha} + \Gamma_{\alpha\beta}\frac{\partial \bar{x}^i}{\partial p_\beta}\right) = \delta^i_j. \tag{4.11}$$

Therefore

$$\left|\frac{\partial \bar{x}^i}{\partial x^\alpha} + \Gamma_{\alpha\beta}\frac{\partial \bar{x}^i}{\partial p_\beta}\right| \neq 0$$

and equations (4.9) determine $\bar{\Gamma}_{ij}$ uniquely. From (4.10) and the inverse of (4.9) we have

$$(\bar{\Gamma}_{ij} - \bar{\Gamma}_{ji})\left(\frac{\partial \bar{x}^j}{\partial x^\alpha} + \Gamma_{\alpha\beta}\frac{\partial \bar{x}^j}{\partial p_\beta}\right) = 0$$

consequently $\bar{\Gamma}_{ij} = \bar{\Gamma}_{ji}$. (See note at end of paper).

Because of (4.7) and (4.8) we have

$$\begin{aligned}\bar{\lambda}^i &= \lambda^\alpha\left(\frac{\partial \bar{x}^i}{\partial x^\alpha} + \Gamma_{\alpha\beta}\frac{\partial \bar{x}^i}{\partial p_\beta}\right) + \lambda_\alpha\frac{\partial \bar{x}^i}{\partial p_\alpha} \\ \bar{\lambda}_i &= \lambda_\alpha\left(\frac{\partial x^\alpha}{\partial \bar{x}^i} + \bar{\Gamma}_{ij}\frac{\partial x^\alpha}{\partial \bar{p}_j}\right)\end{aligned} \tag{4.12}$$

and we accordingly call λ^A *the vector components* of the configuration Λ^A. We may note that any set of functions λ^A obeying (4.11) determine a unique configuration by means of (4.6) and that for two contravariant configurations

$$\Lambda_\alpha M^\alpha - \Lambda^\alpha M_\alpha = \lambda_\alpha \mu^\alpha - \lambda^\alpha \mu_\alpha \tag{4.13}$$

is a scalar.

Similarly the vector components of a covariant configuration Λ_A are defined by

$$\lambda^\alpha = \Lambda^\alpha \qquad \lambda_\alpha = \Lambda_\alpha + \Gamma_{\alpha\beta}\,\Lambda^\beta \tag{4.14}$$

with the law of transformation

$$\begin{aligned} \bar\lambda^i &= \lambda^\alpha \left(\frac{\partial \bar x^i}{\partial x^\alpha} + \Gamma_{\alpha\beta}\frac{\partial \bar x^i}{\partial p_\beta}\right) - \lambda_\alpha \frac{\partial \bar x^i}{\partial p_\alpha} \\ \bar\lambda_i &= \lambda_\alpha \left(\frac{\partial x^\alpha}{\partial \bar x^i} + \bar\Gamma_{ij}\frac{\partial x^\alpha}{\partial \bar p_j}\right). \end{aligned} \tag{4.15}$$

5. **The operators $X_\alpha f$ and $P^\alpha f$.** We define two operators by the equations

$$X_\alpha f(x, p) = \frac{\partial f}{\partial x^\alpha} + \Gamma_{\alpha\beta}\frac{\partial f}{\partial p_\beta}, \qquad P^\alpha f = \frac{\partial f}{\partial p_\alpha} \tag{5.1}$$

and similarly

$$\bar X_i \bar f(\bar x, \bar p) = \frac{\partial \bar f}{\partial \bar x^i} + \bar\Gamma_{ij}\frac{\partial \bar f}{\partial \bar p_j}, \qquad \bar P^i \bar f = \frac{\partial \bar f}{\partial \bar p_i}. \tag{5.2}$$

In this notation the fundamental relations (4.9) and (4.10) are

$$\bar\Gamma_{ij}\,\bar X_\alpha \bar x^j = X_\alpha \bar p_i, \qquad \Gamma_{\alpha\beta}\,\bar X_i x^\beta = \bar X_i p_\alpha \tag{5.3}$$

and from (4.11)

$$X_\alpha \bar x^i \cdot \bar X_i x^\beta = \delta^\beta_\alpha, \qquad X_\alpha \bar x^i \cdot \bar X_j x^\alpha = \delta^i_j. \tag{5.4}$$

In consequence of (1.8) we have

$$\begin{aligned} &P^\alpha \bar x^i + \bar P^i x^\alpha = 0, \qquad P^\alpha \bar x^i\, X_\alpha \bar x^j = P^\alpha \bar x^j\, X_\alpha \bar x^i \\ &P^\alpha \bar p_i\, X_\alpha \bar x^j - P^\alpha \bar x^j\, X_\alpha \bar p_i = \delta^j_i, \qquad P^\alpha \bar p_i\, X_\alpha \bar p_j = P^\alpha \bar p_j\, X_\alpha \bar p_i \end{aligned} \tag{5.5}$$

which are generalizations of (1.11).

The fundamental equations for contravariant and covariant vectors under a homogeneous contact transformation, namely (4.7) and (4.9), take the form

$$\bar\lambda^i = \lambda^\alpha\, X_\alpha \bar x^i + \lambda_\alpha\, P^\alpha \bar x^i, \qquad \bar\lambda_i = \bar X_i x^\alpha \cdot \lambda^\alpha. \tag{5.6}$$

For the commutators of the operators we have

$$\begin{aligned} (X_\alpha, X_\beta) f = (X_\alpha X_\beta - X_\beta X_\alpha) f &= (X_\alpha \Gamma_{\beta\gamma} - X_\beta \Gamma_{\alpha\gamma})\, P^\gamma f \\ &\equiv \Gamma_{\gamma,\alpha\beta}\, P^\gamma f \end{aligned} \tag{5.7}$$

$$(P^\alpha, X_\beta) f = (P^\alpha X_\beta - X_\beta P^\alpha) f = \frac{\partial \Gamma_{\beta\gamma}}{\partial \gamma}, \qquad (P^\alpha, P^\beta) f = 0. \tag{5.8}$$

From (5.7) we have

$$\Gamma_{\gamma,\alpha\beta} = \frac{\partial \Gamma_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial \Gamma_{\alpha\gamma}}{\partial x^\beta} + \Gamma_{\alpha\delta}\frac{\partial \Gamma_{\beta\gamma}}{\partial p_\delta} - \Gamma_{\beta\delta}\frac{\partial \Gamma_{\alpha\gamma}}{\partial p_\delta} \tag{5.9}$$

from which it follows that

$$\Gamma_{\gamma,\alpha\beta} + \Gamma_{\gamma,\beta\alpha} = 0, \qquad \Gamma_{\alpha,\beta\gamma} + \Gamma_{\beta,\gamma\alpha} + \Gamma_{\gamma,\alpha\beta} = 0. \tag{5.10}$$

From (5.3) and (1.8) we obtain

$$\bar{X}_i \bar{f} = \bar{X}_i x^\alpha \cdot X_\alpha f \tag{5.11}$$

and

$$\bar{P}^i \bar{f} = X_\alpha \bar{x}^i \cdot P^\alpha f - P^\alpha \bar{x}^i \cdot X_\alpha f. \tag{5.12}$$

From these relations we have

$$\bar{P}^i \bar{X}_j \bar{f} = X_\alpha f \cdot \bar{P}^i \bar{X}_j x^\alpha + \bar{X}_j x^\alpha \, (X_\beta \bar{x}^i \, P^\beta x_\alpha f - P^\beta \bar{x}^i \, X_\beta X_\alpha f) \tag{5.13}$$

$$\bar{X}_j \bar{P}^i \bar{f} = \bar{X}_j x^\beta \cdot X_\beta \, (X_\alpha \bar{x}^i \, P^\alpha f - P^\gamma \bar{x}^i X_\gamma f). \tag{5.14}$$

Substituting in

$$(\bar{P}^i, \bar{X}_j) \bar{f} = \frac{\partial \bar{\Gamma}_{jk}}{\partial \bar{p}_i} \bar{P}^k \bar{f},$$

we obtain an equation of the form $AX_\alpha f + BP^\gamma f = 0$. Since this must hold for any f, and since $X_\alpha f$ and $P^\alpha f$ are independent as follows from (5.1), we must have $A = B = 0$; this gives the two equations

$$\bar{P}^i \bar{X}_j x^\alpha + \bar{X}_j x^\beta \cdot X_\beta P^\alpha \bar{x}^i + \frac{\partial \bar{\Gamma}_{jk}}{\partial \bar{p}_i} P^\alpha \bar{x}^k = 0 \tag{5.15}$$

$$X_\beta X_\alpha \bar{x}^i = \frac{\partial \Gamma_{\alpha\beta}}{\partial p_\gamma} X_\gamma \bar{x}^i - \frac{\partial \bar{\Gamma}_{jk}}{\partial \bar{p}_i} X_\alpha \bar{x}^j \, X_\beta \bar{x}^k + \Gamma_{\beta,\alpha\gamma} \, P^\gamma \bar{x}^i. \tag{5.16}$$

From (5.11) we have $\bar{X}^i \bar{X}_i \bar{f} = \bar{X}_j \bar{X}_i x^\alpha \cdot X_\alpha f + \bar{X}_i x^\alpha \bar{X}_j x^\beta - X_\beta X_\alpha f$. Substituting in $(\bar{X}_j, \bar{X}_i) \bar{f} = \bar{\Gamma}_{k,ji} \bar{P}^k \bar{f}$ and making use of (5.7), (5.16) and the inverse of the latter, we get

$$(\bar{\Gamma}_{i,jk} \, X_\gamma \bar{x}^i - \Gamma_{\gamma,\alpha\beta} \, \bar{X}_j x^\alpha \bar{X}_k x^\beta) \, P^\gamma f = 0.$$

Since this holds for any f, we have in consequence of (5.4)

$$\bar{\Gamma}_{i,jk} = \Gamma_{\alpha,\beta\gamma} \, \bar{X}_i x^\alpha \cdot \bar{X}_j x^\beta \cdot \bar{X}_k x^\gamma \tag{5.17}$$

and consequently $\Gamma_{\alpha,\beta\gamma}$ are the components of a covariant tensor in V_n skew-symmetric in the indices β and γ.

For a configuration of the second order of components $A^{\alpha\beta}$, $A^\alpha{}_\beta$, $A_\beta{}^\alpha$, $A_{\alpha\beta}$ we have analogously to (4.6)

$$a^{\alpha\beta} = A^{\alpha\beta}, \qquad a^\alpha{}_\beta = a^\alpha{}_\beta - \Gamma_{\beta\gamma} A^{\alpha\gamma}, \qquad a_\beta{}^\alpha = A_\beta{}^\alpha - \Gamma_{\beta\gamma} A^{\gamma\alpha}$$
$$a_{\alpha\beta} = A_{\alpha\beta} - \Gamma_{\alpha\gamma} \, A^\gamma{}_\beta - \Gamma_{\beta\gamma} \, A_\alpha{}^\gamma + \Gamma_{\alpha\gamma} \, \Gamma_{\beta\delta} \, A^{\gamma\delta} \tag{5.18}$$

where $a^{\alpha\beta}$, $a^\alpha{}_\beta$, $a_\beta{}^\alpha$, $a_{\alpha\beta}$ are components of tensors in V_n which under a contact transformation are related as follows:

$$
\begin{aligned}
\bar{a}^{ij} &= (a^{\alpha\beta} X_\beta \bar{x}^j + a^\alpha{}_\beta P^\beta \bar{x}^j) X_\alpha \bar{x}^i + (a_\alpha{}^\beta X_\beta \bar{x}^j + a_{\alpha\beta} P^\beta \bar{x}^j) P^\alpha \bar{x}^i \\
\bar{a}^i{}_j &= (a^\alpha{}_\beta X_\alpha \bar{x}^i + a_{\alpha\beta} P^\alpha \bar{x}^i) \bar{X}_j x^\beta \\
\bar{a}_j{}^i &= (a_\beta{}^\alpha X_\alpha \bar{x}^i + a_{\beta\alpha} P^\alpha \bar{x}^i) \bar{X}_j x^\beta \\
\bar{a}_{ij} &= a_{\alpha\beta} \bar{X}_i x^\alpha \cdot \bar{X}_j x^\beta.
\end{aligned} \tag{5.19}
$$

We call a^{AB} the *tensor components* of the configuration A^{AB}. In particular, if we take

$$
a^\alpha{}_\beta = -\, a_\beta{}^\alpha = \delta^\alpha_\beta \qquad a_{\alpha\beta} = p_\alpha p_\beta \tag{5.20}
$$

we find that the corresponding a's are of similar forms and

$$
\bar{a}^{ij} = a^{\alpha\beta} X_\alpha \bar{x}^i X_\beta \bar{x}^j. \tag{5.21}
$$

For a mixed configuration $A^A{}_B$ the vector components are

$$
\begin{gathered}
a^{\alpha\beta} = A^{\alpha\beta}, \qquad a^\alpha{}_\beta = A^\alpha{}_\beta + \Gamma_{\beta\gamma} A^{\alpha\gamma}, \qquad a_\beta{}^\alpha = A_\beta{}^\alpha - \Gamma_{\beta\gamma} A^{\gamma\alpha}, \\
a_{\alpha\beta} = A_{\alpha\beta} + \Gamma_{\beta\gamma} A_\alpha{}^\gamma - \Gamma_{\alpha\gamma} A^\gamma{}_\beta - \Gamma_{\alpha\gamma}\Gamma_{\beta\delta} A^{\gamma\delta}
\end{gathered} \tag{5.22}
$$

with the component transformation

$$
\begin{aligned}
\bar{a}^{ij} &= (a^{\alpha\beta} X_\beta \bar{x}^j - a^\alpha{}_\beta P^\beta \bar{x}^j) X_\alpha \bar{x}^i + (a_\alpha{}^\beta X_\beta \bar{x}^j - a_{\alpha\beta} P^\beta \bar{x}^j) P^\alpha \bar{x}^i \\
\bar{a}^i{}_j &= (a^\alpha{}_\beta X_\alpha \bar{x}^i + a_{\alpha\beta} P^\alpha \bar{x}^i) \bar{X}_j x^\beta \\
\bar{a}_j{}^i &= (a_\beta{}^\alpha X_\alpha \bar{x}^i - a_{\beta\alpha} P^\alpha \bar{x}^i) \bar{X}_j x^\beta \\
\bar{a}_{ij} &= a_{\alpha\beta} \bar{X}_i x^\alpha \bar{X}_j x^\beta.
\end{aligned} \tag{5.23}
$$

Similarly for a covariant configuration A_{AB} we have the tensor components

$$
\begin{gathered}
a^{\alpha\beta} = A^{\alpha\beta}, \qquad a^\alpha{}_\beta = A^\alpha{}_\beta + \Gamma_{\beta\gamma} A^{\alpha\gamma}, \qquad a_\beta{}^\alpha = A_\beta{}^\alpha + \Gamma_{\beta\gamma} A^{\gamma\alpha} \\
a_{\alpha\beta} = A_{\alpha\beta} + \Gamma_{\alpha\gamma} A^\gamma{}_\beta + \Gamma_{\beta\gamma} A_\alpha{}^\gamma + \Gamma_{\alpha\gamma}\Gamma_{\beta\delta} A^{\gamma\delta}.
\end{gathered} \tag{5.24}
$$

6. **Covariant differentiation.** If λ^A are the vector components of a contravariant configuration Λ^A, the quantities $X_\alpha \lambda^A$, $P^\alpha \lambda^A$ do not have tensor character under contact transformations. In order to obtain invariant configurations from a given one by differentiation, we define covariant differentiation in terms of the components $\overset{A}{l}_{BC}$ of a linear connection assigned to V_n. If we denote the operators X_α and P^α symbolically by $\dfrac{\partial}{\partial s^\alpha}$ and $\dfrac{\partial}{\partial s_\alpha}$, we define covariant differentiation of λ^A by

$$
\lambda^A{}_{,B} = \frac{\partial \lambda^A}{\partial s^B} + \lambda^C \overset{A}{l}_{CB} \tag{6.1}
$$

which is the symbolic expression for the four definitions

$$
\begin{aligned}
\lambda^{\alpha}{}_{;\beta} &= X_\beta \lambda^\alpha + \lambda^\gamma \overset{\alpha}{l}_{\gamma\beta} + \lambda_\gamma \overset{\alpha}{l}{}^{\gamma}{}_{\beta}, \\
\lambda^{\alpha;\beta} &= P^\beta \lambda^\alpha + \lambda^\gamma \overset{\alpha}{l}_{\gamma}{}^{\beta} + \lambda_\gamma \overset{\alpha}{l}{}^{\gamma\beta}, \\
\lambda_\alpha{}^{;\beta} &= P^\beta \lambda_\alpha + \lambda^\gamma \overset{\alpha}{l}_{\gamma}{}^{\beta} + \lambda_\gamma \overset{\alpha}{l}{}^{\gamma\beta}, \\
\lambda_{\alpha;\beta} &= X_\beta \lambda_\alpha + \lambda^\gamma \overset{\alpha}{l}_{\gamma\beta} + \lambda_\gamma \overset{\alpha}{l}{}^{\gamma}{}_{\beta}.
\end{aligned} \tag{6.2}
$$

We seek the conditions connecting the l's and the $\bar{l}$'s in a new system $\bar{x}$, $\bar{p}$ so that $\lambda^{\alpha;\beta}$, $\lambda^{\alpha}{}_{;\beta}$, $\lambda_\alpha{}^{;\beta}$, $\lambda_{\alpha;\beta}$ are four tensors determining a mixed configuration of the second order. In accordance with (5.23), we must have

$$\bar{\lambda}_{i;j} = \bar{X}_i x^\alpha \cdot \bar{X}_j x^\beta \lambda_{\alpha;\beta}.$$

In accordance with (5.6) and (5.11) we have

$$X_\beta \lambda_\alpha = X_\beta X_\alpha \bar{x}^i \cdot \bar{\lambda}_i + X_\alpha \bar{x}^i X_\beta \bar{x}^j \bar{X}_j \bar{\lambda}_i \tag{6.3}$$

and since (6.3) must hold for any $\bar{\lambda}^A$, we obtain

$$
\begin{aligned}
X_\beta X_\alpha \bar{x}^i &= \overset{i}{\bar{l}}_{jk} X_\alpha \bar{x}^j X_\beta \bar{x}^k - \overset{\alpha}{l}{}^{\gamma}{}_{\beta} X_\gamma \bar{x}^i + \overset{\alpha}{l}_{\gamma\beta} P^\gamma \bar{x}^i \\
\overset{i}{\bar{l}}_{jk} &= \overset{\alpha}{l}_{\beta\gamma} \bar{X}_i x^\alpha \bar{X}_j x^\beta \bar{X}_k x^\gamma.
\end{aligned} \tag{6.4}
$$

For the third of (6.2) we must have

$$\bar{\lambda}_i{}^{;j} = \lambda_\alpha{}^{;\beta} \bar{X}_i x^\alpha X_\beta \bar{x}^j - \lambda_{\alpha;\beta} \bar{X}_i x^\alpha P^\beta \bar{x}^j.$$

In accordance with (5.12) we have

$$\bar{P}^j \bar{\lambda}_i = \bar{P}^j \bar{X}_i x^\alpha \cdot \lambda_\alpha + \bar{X}_i x^\alpha (X_\beta \bar{x}^j P^\beta \lambda_\alpha - P^\beta \bar{x}^j X_\beta \lambda_\alpha) \tag{6.5}$$

and we obtain the two sets of equations

$$
\begin{aligned}
\bar{P}^j \bar{X}_i x^\gamma &= \bar{X}_i x^\alpha \Big(\overset{\gamma\beta}{l}_{\alpha} X_\beta \bar{x}^j - \overset{\gamma}{l}_{\alpha\beta} P^\beta \bar{x}^j \Big) - \overset{kj}{\bar{l}}_{i} \bar{X}_k x^\gamma + \overset{j}{\bar{l}}_{ik} \bar{P}^k x^\gamma \\
\overset{j}{\bar{l}}_{i}{}^{k} &= \overset{\beta\gamma}{l}_{\alpha} \bar{X}_i x^\alpha \bar{X}_j x^\beta X_\gamma \bar{x}^k - \overset{\gamma}{l}_{\alpha\beta} \bar{X}_i x^\alpha \bar{X}_j x^\beta P^\gamma \bar{x}^k.
\end{aligned} \tag{6.6}
$$

Proceeding in like manner with the quantities $\lambda^{\alpha}{}_{;\beta}$ and $\lambda^{\alpha;\beta}$ in equations of the form (5.23), we get the two sets of equations

$$
\begin{aligned}
X_\beta X_\alpha \bar{x}^i &= \overset{\gamma}{l}_{\alpha\beta} X_\gamma \bar{x}^i - \overset{i}{\bar{l}}_{jk} X_\alpha \bar{x}^j X_\beta \bar{x}^k + \overset{\gamma}{l}_{\alpha\beta} P^\gamma \bar{x}^i \\
X_\beta P^\alpha \bar{x}^i &= \overset{\gamma}{l}{}^{\alpha}{}_{\beta} X_\gamma \bar{x}^i + \overset{\alpha}{l}_{\gamma}{}^{\beta} P^\gamma \bar{x}^i - \Big(\overset{i}{\bar{l}}{}^{k}_{j} \bar{X}_k x^\alpha + \overset{i}{\bar{l}}_{kj} P^\alpha \bar{x}^k \Big) X_\beta \bar{x}^j
\end{aligned} \tag{6.7}
$$

and

$$P^\beta P^\alpha \bar{x}^i = \overset{\gamma}{l}{}^{\alpha\beta} X_\gamma \bar{x}^i + \underset{\gamma}{l}{}^{\alpha\beta} P^\gamma \bar{x}^i - \left(\overset{i}{l}{}^k_j \bar{X}_k x^\alpha + \overset{i}{l}_{kj} P^\alpha \bar{x}^k\right) P^\beta \bar{x}^j - \left(\overset{i}{l}{}^{kj} \bar{X}_k x^\alpha + \overset{i}{l}{}^{kj} P^\alpha \bar{x}^k\right) \bar{X}_j x^\beta$$

(6.8)

$$X_\beta \bar{x}^j \cdot P^\beta X_\alpha \bar{x}^i - P^\beta \bar{x}^j \cdot X_\beta X_\alpha \bar{x}^i = \left(\overset{\gamma}{l}_\alpha{}^\beta X_\beta \bar{x}^j - \overset{\gamma}{l}_{\alpha\beta} P^\beta \bar{x}^j\right) X_\gamma \bar{x}^i + \left(\underset{\gamma}{l}_\alpha{}^\beta X_\beta \bar{x}^j - \underset{\gamma}{l}_{\alpha\beta} P^\beta \bar{x}^j\right) P^\gamma \bar{x}^i - \overset{i}{l}{}^j X_\alpha \bar{x}^k.$$

The eight sets of equations (6.4), (6.6), (6.7) and (6.8) are just sufficient to determine uniquely the components $\overset{I}{l}_{JK}$ in terms of $\overset{A}{l}_{BC}$. However, we are not free to choose the $\overset{A}{l}_{BC}$ arbitrarily because of (5.15), (5.16) and the commutator conditions (5.7) and the second of (5.8). From the first of (6.4) and (5.7) we get

$$\left(\underset{j}{l}{}^i_k - \underset{k}{l}{}^i_j\right) X_\alpha \bar{x}^j X_\beta \bar{x}^k = \left(\underset{\alpha}{l}{}^\gamma_\beta - \underset{\beta}{l}{}^\gamma_\alpha\right) X_\gamma \bar{x}^i - \left(\underset{\alpha}{l}_{\gamma\beta} - \underset{\beta}{l}_{\gamma\alpha} + \Gamma_{\gamma,\alpha\beta}\right) P^\gamma \bar{x}^i. \tag{6.9}$$

From the first of (6.6), the second of (6.7) and (5.15) we get

$$\left(\overset{i}{l}{}^k_j + \underset{j}{l}{}^{ki}\right) \bar{X}_k x^\alpha = \left(\overset{\gamma}{l}{}^\alpha_\beta + \underset{\beta}{l}{}^{\alpha\gamma}\right) \bar{X}_j x^\beta X_\gamma \bar{x}^i + \left(\underset{\gamma}{l}{}^\alpha_\beta - \underset{\beta}{l}{}^\alpha_\gamma\right) \bar{X}_j x^\beta P^\gamma \bar{x}^i + \left(\frac{\partial \bar{\Gamma}_{jk}}{\partial \bar{p}_i} - \overset{i}{l}_{kj} - \underset{j}{l}{}^i_k\right) P^\alpha \bar{x}^k. \tag{6.10}$$

Since the first of (6.8) must be symmetric in α and β we have

$$\left(\overset{\gamma}{l}{}^{\alpha\beta} - \overset{\gamma}{l}{}^{\beta\alpha}\right) X_\gamma \bar{x}^i + \left(\underset{\gamma}{l}{}^{\alpha\beta} - \underset{\gamma}{l}{}^{\beta\alpha}\right) P^\gamma \bar{x}^i = \left(\overset{i}{l}{}^{jk} - \overset{i}{l}{}^{kj}\right) \bar{X}_j x^\alpha \bar{X}_k x^\beta + \left(\overset{i}{l}_{jk} - \overset{i}{l}_{kj}\right) P^\alpha \bar{x}^j P^\beta \bar{x}^k + \left(\overset{i}{l}{}^k_j - \overset{i}{l}{}^k_j\right) (\bar{X}_k x^\alpha P^\beta \bar{x}^j - \bar{X}_k x^\beta P^\alpha \bar{x}^j). \tag{6.11}$$

From the first of (6.4) and (6.7) we obtain by subtraction

$$\left(\overset{\gamma}{l}_{\alpha\beta} + \underset{\alpha}{l}{}^\gamma_\beta\right) X_\gamma \bar{x}^i - \left(\overset{i}{l}_{jk} + \underset{j}{l}{}^i_k\right) X_\alpha \bar{x}^j X_\beta \bar{x}^k + \left(\underset{\gamma}{l}_{\alpha\beta} - \underset{\alpha}{l}_{\gamma\beta}\right) P^\gamma \bar{x}^i = 0. \tag{6.12}$$

If the first of (6.7) and the inverse of the first of (6.6) are substituted in the second of (6.8) we obtain

$$\left(\overset{i}{l}{}^j_k + \underset{k}{l}{}^{ij}\right) X_\gamma \bar{x}^k + \left(\underset{k}{l}{}^i_l + \overset{i}{l}_{kl}\right) X_\beta \bar{x}^l P^\gamma \bar{x}^k P^\beta \bar{x}^j - \left(\overset{\alpha}{l}_\gamma{}^\beta + \underset{\gamma}{l}{}^{\alpha\beta}\right) X_\alpha \bar{x}^i X_\beta \bar{x}^j + \left(\underset{\gamma}{l}_\alpha{}^\beta - \underset{\alpha}{l}_\gamma{}^\beta\right) X_\beta \bar{x}^j P^\alpha \bar{x}^i = 0. \tag{6.13}$$

From (6.6), (6.7) and (6.8) we obtain

$$\left(\overset{i}{l}_{jk} + \overset{i}{l}_{k j} - \frac{\partial \bar{\Gamma}_{jk}}{\partial \bar{p}_i}\right) X_\alpha \bar{x}^j X_\beta \bar{x}^k = \left(\overset{\gamma}{l}_{\alpha\beta} + \overset{\gamma}{l}_{\beta\alpha} - \frac{\partial \Gamma_{\alpha\beta}}{\partial p_\gamma}\right) X_\gamma \bar{x}^i + \left(\overset{\gamma}{l}_{\alpha\beta} - \overset{\gamma}{l}_{\beta\alpha} + \Gamma_{\alpha\beta\gamma}\right) P_\gamma \bar{x}^i. \tag{6.14}$$

7. **Symmetry conditions on the linear connection.** From (5.18), (5.22) and (5.24) it follows that the tensor components of the three fundamental numerical configurations ϵ^{AB}, ϵ_{AB} and δ^A_B have the same respective forms as the components of the configuration. From §5 it follows that if $\Lambda_A = \epsilon_{AB}\Lambda^B$, the tensor components of these *associate* covariant and contravariant configurations are in the relation

$$\lambda_A = \epsilon_{AB}\lambda^B. \tag{7.1}$$

We define the covariant derivatives of λ_A by equations analogous to (6.1), namely

$$\lambda_{A;B} = \frac{\partial \lambda_A}{\partial s^B} - \lambda_C \overset{C}{l}_{AB}. \tag{7.2}$$

Extensions of the definitions (6.1) and (7.2) to tensor components of configurations of higher order are made as in ordinary tensor analysis. We require that the covariant derivative of an associate configuration be the associate of the covariant derivative of the original configuration. In particular, for (7.1) we must have

$$\lambda_{A;B} = \epsilon_{AC}\lambda^C{}_{;B}$$

but from (7.2) we have

$$\lambda_{A;B} = \epsilon_{AC}\lambda^C{}_{;B} + \lambda^C \epsilon_{AC;B}.$$

Consequently the above requirement is equivalent to

$$\epsilon_{AB;C} = 0 \tag{7.3}$$

Since $\delta^A_{B;C} = 0$, it follows from (3.6) that

$$\epsilon^{AB}{}_{;C} = 0. \tag{7.4}$$

In order that (7.3) be satisfied, it is necessary and sufficient that the components of the connection satisfy the six sets of relations

$$\overset{\alpha}{l}{}^{\beta}_{C} = \overset{\beta}{l}{}^{\alpha}_{C}, \qquad \overset{\alpha}{l}_{\beta C} = -\overset{\alpha}{l}_{\beta} C, \qquad \overset{\alpha}{l}_{\beta C} = \overset{\beta}{l}_{\alpha C}. \tag{7.5}$$

If λ^A and μ^A are the vector components of two contravariant configurations of the first order, we have shown in (4.13) that $\varphi = \lambda_\alpha \mu^\alpha - \lambda^\alpha \mu_\alpha$ is a scalar. The requirement that

$$\varphi_{;\alpha} = X_\alpha \varphi, \qquad \varphi^{;\alpha} = P^\alpha \varphi$$

is equivalent to (7.5).

In view of (7.5) the conditions (6.12) and (6.13) are satisfied identically. We impose further symmetry conditions on the linear connection to satisfy the other conditions in §6. Since $\overset{\alpha}{l}_{\beta\gamma}$ and $\Gamma_{\alpha,\beta\gamma}$ each obey the same linear homogeneous law of transformation, we take

$$\overset{\alpha}{l}_{\gamma\beta} - \overset{\beta}{l}_{\gamma\alpha} = \Gamma_{\gamma,\beta\alpha}. \tag{7.6}$$

From (7.6) and (6.14) it follows that we may take

$$\overset{\alpha}{l}_{\beta\gamma} + \overset{\alpha}{l}_{\beta}{}^{\gamma} = \frac{\partial\Gamma_{\beta\gamma}}{\partial p_\alpha} \tag{7.7}$$

as an invariant condition. From (7.7) and (7.5) we have

$$\overset{\alpha}{l}_{\beta\gamma} = \overset{\alpha}{l}_{\gamma\beta}, \tag{7.8}$$

in consequence of which and (7.6) the conditions (6.9) are identically satisfied. In consequence of (7.7), (7.8) and (7.5) we may take

$$\overset{\alpha}{l}{}^{\beta}{}_{\gamma} = \overset{\alpha}{l}_{\gamma}{}^{\beta}, \tag{7.9}$$

from which it follows by (7.5) that

$$\overset{\alpha}{l}{}^{\beta\gamma} = \overset{\alpha}{l}{}^{\gamma\beta}. \tag{7.10}$$

Making use of the above relations and (6.11) we may take

$$\overset{\alpha}{l}{}^{\beta\gamma} = \overset{\alpha}{l}{}^{\gamma\beta} \tag{7.11}$$

as a result of which and (7.5) $\overset{\alpha}{l}{}^{\beta\gamma}$ is symmetric in all indices.

All these conditions may be combined into the following definition:

The components of linear connection $\overset{A}{l}_{BC}$ are chosen subject to the invariant conditions

$$\epsilon^{AK}\overset{B}{l}_{KC} = \epsilon^{BK}\overset{A}{l}_{KC} \tag{7.12}$$

$$\overset{\alpha}{l}_{BC} = \overset{\alpha}{l}_{CB} \tag{7.13}$$

with the addition of (7.6), (7.7) and (7.10).

8. **Curvature; the Ricci identity.** In ordinary tensor analysis covariant differentiation is not commutative except for scalar functions. If $t^{\cdots}_{\cdots}$ are the components of a tensor, $t^{\cdots}_{\cdots,i,j} - t^{\cdots}_{\cdots,j,i}$ is expressible linearly in terms of the components of the tensor, the coefficients being the components of the curvature tensor of the connection employed in the covariant differentiation. We shall show that this is the case with our linear connection provided that the symmetry

conditions of §7 are satisfied. In fact the above requirement is equivalent to the full set of symmetry relations.

Let λ^A be the vector components of a configuration of the first order; then

$$\lambda^A_{;B;C} = \frac{\partial \lambda^A_{;B}}{\partial s^C} + \lambda^D_{;B} \overset{A}{l}_{DC} - \lambda^A_{;D} \overset{D}{l}_{BC}$$

$$= \frac{\partial^2 \lambda^A}{\partial s^C \partial s^B} + \lambda^D \frac{\partial \overset{A}{l}_{DB}}{\partial s^C} - \lambda^E \overset{A}{l}_{DB} \overset{D}{l}_{EC}$$

$$+ \lambda^D_{;C} \overset{A}{l}_{DB} + \lambda^D_{;B} \overset{A}{l}_{DC} - \lambda^A_{;D} \overset{D}{l}_{BC}.$$

Therefore

$$\lambda^A_{;B;C} - \lambda^A_{;C;B} = \frac{\partial^2 \lambda^A}{\partial s^C \partial s^B} - \frac{\partial^2 \lambda^A}{\partial s^B \partial s^C} - \lambda^A_{;D} \left(\overset{D}{l}_{BC} - \overset{D}{l}_{CB} \right)$$

(8.1)

$$+ \lambda^D \left(\frac{\partial \overset{A}{l}_{DB}}{\partial s^C} - \frac{\partial \overset{A}{l}_{DC}}{\partial s^B} + \overset{E}{l}_{DB} \overset{A}{l}_{EC} - \overset{E}{l}_{DC} \overset{A}{l}_{EB} \right).$$

We have to consider three possible cases of which we shall develop only one viz. $B = \beta, C = \gamma$. Here

$$(X_\gamma X_\beta - X_\beta X_\gamma)\lambda^A - \lambda^A_{;D} \left(\overset{D}{l}_{\beta\gamma} - \overset{D}{l}_{\gamma\beta} \right) = \Gamma_{\delta\gamma\beta} P^\delta \lambda^A + \Gamma_{\delta\beta\gamma} \left(P^\delta \lambda^A + \lambda^D \overset{A}{l}_D{}^\delta \right)$$

$$= \lambda^D \overset{A}{l}_{DE} \left(\overset{E}{l}_{\beta\gamma} - \overset{E}{l}_{\gamma\beta} \right)$$

because of (7.6) and (7.13). Similar results are obtainable for the cases $B = \beta$, $C = n + \gamma$ and $B = n + \beta$, $C = n + \gamma$; we therefore have

$$\lambda^A_{;B;C} - \lambda^A_{;C;B} = \lambda^D \overset{A}{K}_{DBC} \tag{8.2}$$

where

$$\overset{A}{K}_{DBC} = \frac{\partial \overset{A}{l}_{DB}}{\partial s^C} - \frac{\partial \overset{A}{l}_{DC}}{\partial s^B} + \overset{E}{l}_{DB} \overset{A}{l}_{EC} - \overset{E}{l}_{DC} \overset{A}{l}_{EB} - \overset{A}{l}_{DE} \left(\overset{E}{l}_{BC} - \overset{E}{l}_{CB} \right). \tag{8.3}$$

Equations (8.2) are the analogues of the identity of Ricci and $\overset{A}{K}_{DBC}$ are the tensor components of a configuration of the fourth order that we designate as the curvature of our linear connection. It is easily verifiable that $\overset{A}{K}_{DBC}$ have the same symmetry properties in the indices A and D that $\overset{A}{l}_{DE}$ have.

Since $\Gamma_{\alpha\beta}$ is homogeneous of degree 1 in the p's, the contact frame is completely determined by the linear connection by means of (7.7). It is then not difficult to verify that the vanishing of the curvature tensor is a necessary and sufficient condition that the linear connection be flat, that is, that there exist a homogeneous contact transformation which transforms the given connection $\overset{A}{l}_{BC}$ into $\overset{A}{l}_{BC} = 0$.

9. **Parallelism. Paths.** If by a curve in V_n we understand a set of points given by $x^\alpha = x^\alpha(t)$, we are not in a position to apply a contact transformation to the curve unless we have a way of attaching at each point an elemental hypersurface $p_\alpha(t)$. In order that this be done in an invariant manner we make use of our contact frame and require that the vector components of the configuration $\frac{dx^i}{dt}, \frac{dp_i}{dt}$ be $\frac{dx^i}{dt}, 0$.[5] We therefore assume that $x^i = x^i(t)$, $p_i = p_i(t)$ define a curve, if and only if

$$\frac{dp_i}{dt} = \Gamma_{ij}(x, p)\frac{dx^j}{dt}. \tag{9.1}$$

That this condition is invariant under contact transformations follows from the law of transformation of the contact frame.

We now say that a configuration whose vector components are λ^A, is parallel with respect to a curve if

$$\lambda^A_{;\alpha}\frac{dx^\alpha}{dt} = 0 \tag{9.2}$$

and a configuration field is parallel if

$$\lambda^A_{;\alpha} = 0. \tag{9.3}$$

Equations (9.3) are also invariant under contact transformations since the transformation law of the components $\mu^A{}_\alpha$ does not involve $\mu^{A\alpha}$. In particular we say a curve is a path if the configuration $\frac{dx^\alpha}{dt}, \frac{dp_\alpha}{dt}$ is parallel with respect to itself. This leads to the equations

$$\begin{aligned} \frac{d^2x^\alpha}{dt^2} + \overset{\alpha}{l}_{\beta\gamma}(x, p)\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} &= 0 \\ \frac{dp_\alpha}{dt} = \Gamma_{\alpha\beta}(x, p)\frac{dx^\beta}{dt} & \end{aligned} \tag{9.4}$$

and

$$\underset{\alpha}{l}_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} = 0. \tag{9.5}$$

If we require that there be a path through each element x_0, p_0 in any direction $\left(\frac{dx}{dt}\right)_0$, $\underset{\alpha}{l}_{\beta\gamma} + \underset{\alpha}{l}_{\gamma\beta} = 0$. Hence we assume

$$\underset{\alpha}{l}_{\beta\gamma} = -\tfrac{1}{2}\Gamma_{\alpha,\beta\gamma} \tag{9.6}$$

so that (9.5) are satisfied identically.

[5] There would be no gain in generality if we required the vector components to be $\frac{dx^i}{dt}$, $\varphi(t)p_i$ as by a change of parameter this is reducible to the assumption we are making.

10. **The metric. Extremals.** If λ^A are the vector components of a configuration, its length is given by

$$\lambda^2 = g_{AB}(x, p)\, \lambda^A \lambda^B \tag{10.1}$$

where $g_{AB} = g_{BA}$ are the tensor components of a configuration of the second order and are assumed to be homogeneous of degree zero in the p's. Just as in the case of paths we did not need the full linear connection so in the case of curve length we do not need the full metric tensor. According to (10.1) the length of a curve is given by

$$ds^2 = g_{\alpha\beta}(x, p)\, dx^\alpha\, dx^\beta. \tag{10.2}$$

Let (x_0, p_0) and (x_1, p_1) be joined by a curve

$$x = x(t), \qquad p = p(t), \qquad dp_i = \Gamma_{ij} dx^j, \tag{10.3}$$

and consider the field of curves given by

$$\begin{aligned} \bar{x}^i &= \bar{x}^i(t) + \omega^i(x, p)\delta\sigma \\ \bar{p}_i &= p_i(t) + \omega_i(x, p)\delta\sigma \end{aligned} \tag{10.4}$$

where ω^i and ω_i are so chosen that the points corresponding to the same value of t should constitute a curve; that is, since $\frac{\delta x^i}{\delta\sigma} = \omega^i$ and $\frac{\delta p_i}{\delta\sigma} = \omega_i$ we must have

$$\omega_i = \Gamma_{ij}\omega^j \tag{10.5}$$

Since for a given $\delta\sigma$, $\bar{x}$ and $\bar{p}$ must define a curve, we get from (10.4) and (10.5)

$$d\omega_i = \Gamma_{ij}\, d\omega^j + \omega^k X_k \Gamma_{ij} \cdot dx^j. \tag{10.6}$$

From (10.5) we have

$$d\omega_i = \Gamma_{ij}\, d\omega^j + \omega^k X_j \Gamma_{ik}\, dx^j.$$

Comparing these two results we find

$$\Gamma_{i,jk}\, dx^j\, \delta x^k = 0.^6 \tag{10.7}$$

When we minimize the arc length from P_0 to P_1 we find, as in Riemannian geometry, that the Euler equations become

$$\frac{\partial\varphi}{\partial x^i} + \Gamma_{ij}\frac{\partial\varphi}{\partial p_j} - \frac{d}{dt}\frac{\partial\varphi}{\partial\dot{x}^i} = 0 \tag{10.8}$$

where $\varphi = \frac{ds}{dt}$ and $\dot{x}^i = \frac{dx^i}{dt}$. Since $\frac{\partial}{\partial x^i} + \Gamma_{ij}\frac{\partial}{\partial p_j} = X_i$, if we let the parameter

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[6] The geometrical significance of this set of equations is that if the configuration $(p_i, 0)$ is carried by parallel displacement around the contour x, $x + \delta x$, $x + \delta x + dx$, $x + dx$, the components are unchanged.

be the arc length, the equations of the extremals are found to be

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

$$\frac{dp_i}{ds} = \Gamma_{ij}\frac{dx^j}{ds} \tag{10.9}$$

where the Christoffel symbols are formed out of $g_{\alpha\beta}$ using the operator X_α for $\frac{\partial}{\partial x^\alpha}$.

We thus have, as in §9, a unique extremal determined by the initial conditions $x_0, p_0, \dot{x}_0$. A particular set of curves are given by the differential equations

$$\frac{dx^i}{ds} = g^{ij}(x, p)p_j \qquad \frac{dp_i}{ds} = \Gamma_{ih}g^{hj}p_j . \tag{10.10}$$

The requirement that these curves be extremals gives

$$g^{\gamma\delta}p_\delta \cdot p_\beta \left(X_\gamma g^{\alpha\beta} + P^\beta \Gamma_{\epsilon\gamma} \cdot g^{\alpha\epsilon} + \left\{ \begin{matrix} \alpha \\ \epsilon\gamma \end{matrix} \right\} g^{\beta\epsilon} \right) = 0.$$

Hence as in the case of a linear connection (Cf. (7.7)), if we let

$$P^\alpha \Gamma_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + l_{\gamma\ \beta}^{\ \alpha},$$

we shall have

$$l_{\gamma\ \beta}^{\ \alpha} p_\alpha g^{\gamma\delta} p_\delta = 0.^7 \tag{10.11}$$

More generally if $C(x, p)$ is a scalar function homogeneous of degree one in the p's, $\frac{\partial C}{\partial x^\alpha}, \frac{\partial C}{\partial p_\alpha}$ is a covariant configuration of the first order whose contravariant vector components are $\frac{\partial C}{\partial p_\alpha}$ and $X_\alpha C$; these components will determine a curve

$$\frac{dx^\alpha}{ds} = \frac{\partial C}{\partial p_\alpha}, \qquad \frac{dp_\alpha}{ds} = -\frac{\partial C}{\partial x^\alpha} \tag{10.12}$$

if and only if

$$X_\alpha C = 0. \tag{10.13}$$

Equations (10.13) are a system of partial differential equations whose integrability conditions are $\Gamma_{\alpha,\beta\gamma}P^\alpha C = 0$, $(X_\delta \Gamma_{\alpha,\beta\gamma} - \Gamma_{\delta\alpha}{}^\epsilon \Gamma_{\epsilon,\beta\gamma})P^\alpha C = 0$ etc. Assuming that these conditions can be satisfied and that C is so chosen as to satisfy (10.13), we consider the question whether these curves can be extremals.[8]

[7] The equations (10, 11) mean that the vector $(p_\alpha, 0)$ is parallel with respect to the extremals (10.10).

[8] Cf. *C. G.* §67 where C is taken to be $\sqrt{(g^{ij}p_ip_j)}$.

If this is to be the case we must have

$$X_\beta P^\alpha C \cdot P^\beta C = -\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} P^\beta C \cdot P^\gamma C$$

and since

$$X_\beta P^\alpha C = P^\alpha X_\beta C - P^\alpha \Gamma_{\beta\gamma} \cdot P^\gamma C = -P^\alpha \Gamma_{\beta\gamma} \cdot P^\gamma C$$

the required condition reduces to

$$\left(P^\alpha \Gamma_{\beta\gamma} - \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} \right) P^\beta C \cdot P^\gamma C = 0.$$

NOTE (*added in proofs, October 15, 1936*): In §4 it is shown that if equations (4.9) are consistent they admit a unique solution. It is therefore understood that a V_n with a frame $\Gamma_{\alpha\beta}$ being given, it is subjected to transformations for which (4.9) are consistent.

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THE SUBGROUP OF THE ELEMENTS OF FINITE ORDER OF AN ABELIAN GROUP†

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Every (additively written) Abelian group A determines two characteristic "unmixed" groups: the subgroup $F(A)$ of all the elements of finite order in A and the classgroup $I(A) = A/F(A)$. All the elements of $F(A)$ have finite order and 0 is the only element of finite order in $I(A)$. If conversely all the elements of the Abelian group F have finite order and 0 is the only element of finite order of the Abelian group J, then the direct sum $A = F + J$ satisfies $F(A) = F$ and $I(A) = J$.[1] The structure of A is therefore completely determined by the structure of $F(A)$ and of $I(A)$ if, and only if, $F(A)$ is a direct summand of every group A satisfying $F(A) = F$ and $I(A) = J$.

In the following conditions will be given for the group $F(A)$ to be a direct summand of the group A. In accordance with the above considerations we are not interested in conditions which are dependent on the structure of the whole group A, but only in such conditions which depend on the structure of $F(A)$ and (or) $I(A)$ alone.

The principal results can be found in section 8. and the concepts used for enunciating them in the sections 2 and 3.

1. Subgroups satisfying $nS = S$ for every integer n.

(1;1) *If the subgroup S of the Abelian group A satisfies $nS = S$ for every integer $n \neq 0$, then S is a direct summand of A.*

PROOF:[2] Let T be a greatest subgroup of A such that the intersection of S and T contains only the element 0. Then the subgroup of A which is generated by the elements of S and T is the direct sum $S + T$ of S and T. If x is any element of A, then the subgroup $T(x)$ of A, generated by x and the elements in T, contains only elements of the form $nx + t$ where n is an integer and t an element of T. By the choice of T either x is contained in T or there exists an integer m and an element t' in T such that $mx - t'$ is an element $\neq 0$ in S.

In particular to every element x in A there exists an integer $n \neq 0$ such that $nx = s + t$ is an element in $S + T$. Since $S = nS$, there exists an element s'

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[1] This is not quite correct, since the groups $I(A)$ and J are only isomorphic, but not identical. But here and in the following we identify isomorphic groups wherever that is possible without confusion, in particular always when there exists a "natural" isomorphism between a given group and a class group.

[2] The statement (1;1) is well known. But we prove it here, since, to the authors knowledge, it has not been published before.

in S such that $ns' = s$ and therefore the element $x' = x - s'$ will satisfy: $x' \equiv x \bmod S$ and $nx' = t$. If X is the set of all the elements y such that $y \equiv x \bmod S$ and $ny \equiv 0 \bmod T$ for a certain positive integer n, then X is not vacuous and contains elements w satisfying $vw \equiv 0 \bmod T$ with a smallest positive v. But then there exists, as noted before, an integer u and elements $a \neq 0$ in S, b in T such that $uw = a + b$ provided w is not contained in T. If (u, v) is the greatest common divisor of u and v, then there exist integers h and k such that $(u, v) = hu + kv$ and therefore $(u, v)w = ha + (kvw + hb) \equiv ha \bmod T$. It follows as before that X contains an element w' satisfying $(u, v)\, w' \equiv 0 \bmod T$ and therefore that $v = (u, v)$ i.e. u is a multiple of v. But that leads to a contradiction with $a \neq 0$ and therefore w is contained in T, i.e. x is congruent mod S to an element of T, i.e. $A = S + T$.

2. **Groups without elements of infinite order.**[3] If F is an Abelian group without elements of infinite order, then to every prime number p there belongs the primary-component $C(p, F)$ of F which consists of all those elements of F whose order is a power of p. F is the direct sum of its primary-components $C(p, F)$ and therefore every element x of F is the sum of its uniquely determined primary-components $c(p, x)$ contained in $C(p, F)$ respectively.

Let now B be a primary Abelian group, belonging to the prime number p. Then $p^\omega B$ is the intersection of all the groups $p^i B$ for positive integers i.

(2;1) *If the orders of the elements of $B/p^\omega B$ are bounded, then $p^\omega B = p(p^\omega B)$. $p^\omega B$ is therefore (by (1;1)) a direct summand of B and B a direct sum of $p^\omega B$ and of cyclic groups (of bounded orders).*

(2;2) *If the orders of the elements of $B/p^\omega B$ are not bounded, then there exist elements $a(i)$ in B such that the orders of the elements $b(i) = \sum_{j=0}^{i-1} p^j a(j)$ tend to infinity with i and such that $b(i)$ is an element of lowest order in its class* mod $p^i B$.

Since the primary group B belonging to the prime number p satisfies $qB = B$ for every prime number $q \neq p$, the statement (1;1) implies:

(2;3) *Let $Q(A)$ be the direct sum of the groups $C(p, F(A))$ which satisfy $pC(p, F(A)) = C(p, F(A))$. Then $Q(A)$ is a direct summand of A.*

It is now easy to give an example of a group A such that $F(A)$ is not a direct summand of A.[4] For let B be a primary group such that the orders of its elements are not bounded and such that $p^\omega B = 0$. The sets $K(i)$ for $i = 1, 2, \cdots$ form an L-series of B, if $K(i)$ is a class of B mod $p^i B$ and $K(i) < K(i-1)$ for every i. The intersection of all the classes of an L-series is either vacuous or contains exactly one element, since $p^\omega B = 0$. Therefore it is possible to identify the L-series $p^i B + x$ with the element x of B. If $K(i)$ and $H(i)$ are two L-series, then $K(i) - H(i)$ is also an L-series. The set $\bar{B}$ of all the L-series in B is therefore an Abelian group, containing B, the p-adic closure of B.

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[3] This section is a compilation of mostly well-known facts.

[4] A first example of such a group has been given by F. Levi: *Abel-sche Gruppen mit abzählbaren Elementen*, Hab.-Schrift, Leipzig, 1919.

Since the orders of the elements of B are not bounded, it follows from (2;2) that there exist elements of infinite order in $\bar{B}$. Furthermore the only direct summand of $\bar{B}$ which contains B is $\bar{B}$. Since $F(\bar{B})$ contains B, it follows therefore that $F(\bar{B})$ is not a direct summand of $\bar{B}$.

3. **Groups without elements of finite order.** Suppose that 0 is the only element of finite order in the (Abelian) group[5] J. If p is a prime number, then J is said to be *p-complete*, if $J = pJ \neq 0$. If J is p-complete for every prime number p, then J is *complete*.

By (1;1) every complete subgroup of J is a direct summand of J. Since the joingroup of two complete subgroups is also a complete subgroup it follows that there exists a uniquely determined greatest complete subgroup, provided there exist complete subgroups at all.

(3;1) *Every group J without elements of finite order is contained in one and essentially only one smallest complete group.*

PROOF: Let P be the set of all the pairs (x, n) for x in J and n a positive integer. Define equality in P by $x = (x, 1)$, $(x, n) = (y, m)$ if, and only if, $mx = ny$, and addition by $(x, n) + (y, m) = (mx + ny, mn)$.

This group P is a smallest complete group, containing J, and if P' is another smallest complete group, containing J, then there exists an isomorphism between P and P' which leaves all the elements in J invariant.

If G is a greatest linearly independent subset of the complete group J, (x) the subgroup of all the elements of J which are dependent on x, then J is the direct sum of all the groups (g) for g in G and every group (g) is isomorphic with the additive group of all the rational numbers. As a consequence of these facts and of (3;1) it follows:

(3;2) *If J is a group without elements of finite order, then any two greatest linearly independent subsets of J contain the same number of elements.* This number is the *rank* of J.

The subgroup S of the group J is *closed* if J/S does not contain elements of finite order, i.e. if all the elements of J which are dependent on S are contained in S.

DEFINITION 3;2: *Let J be an Abelian group without elements of finite order. If J is countable, then $D(J) = 1$. If v is any (finite or infinite) positive ordinal, then $D(J) = v$, if $D(J) \not< v$ and if there exists a closed subgroup S of finite rank such that J/S is a direct sum of groups J' with $D(J') < v$.*

Note that there exist groups J such that $D(J)$ is not defined e.g. the group of all the sequences of integers.

(3;3) *Suppose that the group J without elements of finite order contains a closed subgroup S of finite rank such that $D(J/S)$ exists. Then $D(J)$ exists and satisfies $D(J) \leqq D(J/S)$.*

PROOF: If $J^* = J/S$, $D(J^*) = v$, then there exists a closed subgroup T^* of J^*

[5] Then we say that J is a group without elements of finite order.

of finite rank such that $J^{**} = J^*/T^*$ is the direct sum of groups $J^{**}(u)$ satisfying $D(J^{**}(u)) < v$. Let $J^*(u)$ be the subgroup of J^* which corresponds to $J^{**}(u)$, T and $J(u)$ the subgroups of J corresponding to $J^*(u)$ and T^* respectively. Then T is a closed subgroup of finite rank of J and, since $J(u)/T$ and $J^{**}(u)$ are isomorphic, it follows that $D(J(u)/T) < v$. Hence J/T is the direct sum of the groups $J(u)/T$ of smaller D and therefore $D(J) \leqq v$.

The following concepts will be needed later on:

Let P be an (infinite) set of prime numbers. Then the element x is *P-infinite* (in J), if $x \neq 0$ and if for an infinity of prime numbers p in P there exist solutions of the equation $py = x$ in J. If the group J contains a P-infinite element, then J itself is called P-infinite and J is called *almost P-infinite* if there exists a closed subgroup S of finite rank such that J/S is P-infinite.

Similarly: if p is a prime number, then J is called *almost p-infinite* if there exists a closed subgroup S of finite rank such that J/S contains a p-complete subgroup.

If e.g. J is the additive group of all the integer p-adic numbers, then $D(J) = 2$ and J is almost p-infinite but does not contain any p-complete subgroup.

4. Extension of groups.

(4;1) *If A is any Abelian group, $x^* \neq 0$ an element of $A/F(A)$ and p a prime number, then there exists a group A' such that*

(a) $A \leqq A'$,
(b) $F(A) = F(A')$,
(c) *the equation $py^* = x^*$ has a solution y^* in $A'/F(A)$,*
(d) *either $A = A'$ or A'/A is a group of order p.*

Proof: If the equation $py^* = x^*$ has a solution y^* in $A/F(A)$, then $A' = A$ satisfies the conditions (a) – (d). If on the other hand the equation $py^* = x^*$ has no solution y^* in $A/F(A)$, let x be any element in A such that $x^* = F(A) + x$ and let A' denote the group which contains A and an element y such that $py = x$. Then every element of A' has the form $a + cy$ for a in A and $0 \leqq c < p$ where a, c are uniquely determined by the element $a + cy$. Suppose now that n is a positive integer such that $n(a + cy) = 0$. If $c = 0$, then a is an element of $F(A)$. If $c \neq 0$, then c and p are relatively prime and, since $-na = ncy$, it follows that nc and therefore n is a multiple of p, i.e. $n = pm$ and hence $0 = m(pa + cx)$. Therefore $pa + cx$ is (as an element of A) contained in $F(A)$ or $p(-a) \equiv cx \bmod F(A)$. But since p is prime to c, this implies that $py^* = x^*$ has a solution y^* in $A/F(A)$ and this is impossible. Therefore $F(A) = F(A')$, i.e. A' satisfies the conditions (a) – (d).

(4;2) *If A is any Abelian group, then there exists an Abelian group A' such that*

(a) $A \leqq A'$,
(b) $F(A) = F(A')$
(c) *$A'/F(A')$ is complete,* [*or* 0],
(d) *all the elements of A'/A are of finite order.*

PROOF: By well-ordering the elements of $A/F(A)$ and by applying (4:1) successively on all the elements of $A/F(A)$ it follows that there exists a group A^* such that

(a) $A \leqq A^*$,
(b) $F(A) = F(A^*)$,
(c) the equation $py^* = x^*$ has a solution y^* in $A^*/F(A)$ for every element $x^* \neq 0$ in $A/F(A)$ and every prime number p,
(d) A^*/A contains only elements of finite order.

To A^* there exists an analogous group $(A^*)^* = A^{**}$ and so on and the join A' of all these groups satisfies the conditions of (4;2).

(4;3) *Suppose that A is an Abelian group, J a group without elements of finite order and $A/F(A)$ a subgroup of J. Then there exists a group A' such that*

(a) $A \leqq A'$,
(b) $F(A) = F(A')$,
(c) $A'/F(A) = J$ (i.e. there exists an isomorphism between $A'/F(A)$ and J which leaves invariant the elements of $A/F(A)$).

PROOF: By (4;2) there exists a group A'' such that $A \leqq A''$, $F(A) = F(A'')$, $A''/F(A)$ is complete and is the (by (3;1) essentially uniquely determined) smallest complete group which contains $A/F(A)$. Furthermore there exists by (3;1) an essentially uniquely determined smallest complete group C which contains J. Since $A/F(A)$ is a subgroup of J, there exists exactly one smallest complete subgroup S of C which contains $A/F(A)$ which is therefore essentially identical with $A''/F(A)$. By (1;1) S is a direct summand of C, i.e. $C = S + T$. Now it is easy to see that there exists a group A' between A'' and $A'' + T$ which satisfies the conditions (a) – (c).

(4;4) *If F is a group without elements of infinite order, J a group without elements of finite order and $F(A)$ is a direct summand of every group A such that $F(A) = F$ and $A/F(A) = J$, then $F(B)$ is a direct summand of every group B such that $F(B) = F$ and $B/F(B) \leqq J$.*

PROOF: By (4;3) B is contained in a group A such that $F(A) = F(B) = F$ and $A/F(A) = J$. $F(A)$ is therefore from the assumption a direct summand of A, i.e. $A = H + F(A)$. If K denotes the intersection of H and B, then $B = K + F(B)$, since $F(A) = F(B)$.

(4;5) *If F is a group without elements of infinite order, J a group without elements of finite order, and if $F(A)$ is a direct summand of every group A such that $F(A) = C(p, F)$ and $A/F(A) = J$, then $C(p, F(B))$ is a direct summand of every group B such that $F(B) = F$ and $B/F(B) = J$.*

PROOF: If Q is the direct sum of all the $C(q, F)$ with $q \neq p$, $A = B/Q$, then $F(A)$ is essentially $= C(p, F)$, since every class of $F(B)/Q$ contains exactly one element of $C(p, F)$. Furthermore $A/F(A) = J$. $F(A)$ is therefore by the assumption a direct summand of A, i.e. $A = H + F(A)$. Let K be the subgroup of B corresponding to H (such that $Q \leqq K$ and $K/Q = H$). The intersection of K and $C(p, F)$ equals the intersection of Q and $C(p, F)$ and consists

therefore of the 0-element alone. B is generated by the elements in $C(p, F)$ and in K, since they represent all the classes of B/Q. Hence $B = C(p, F) + K = C(p, F(B)) + K$.

5. The first necessary condition.

(5;1) *Suppose that the group J without elements of finite order contains an almost p-complete subgroup. Then J contains two subgroups S, T and elements $e(i)$ such that*

(a) *S is a closed subgroup of finite rank of T,*
(b) *S does not contain any almost p-infinite subgroup,*
(c) *T is generated by the elements $e(i)$ and the elements in S,*
(d) $e(0) \not\equiv 0 \bmod S$, $pe(i) \equiv e(i-1) \bmod S$.

PROOF: If V is an almost p-complete subgroup of J, then there exists a closed subgroup W of V which is of finite rank such that V/W is p-complete. Therefore V contains an element x which is not contained in W and the smallest closed subgroup of V which contains x and W is of finite rank and almost p-infinite The existence of an almost p-infinite subgroup of J implies therefore the existence of an almost p-infinite subgroup of finite rank. Therefore there exists an almost p-infinite subgroup U of smallest (finite) rank. Since U is almost p-infinite, it contains a subgroup S (closed in U) such that U/S is p-complete. The rank of S is smaller (and therefore finite) than the rank of U, since $U/S \neq 0$, and therefore S cannot contain any almost p-infinite subgroup. If $e(0)$ is an element of U which is not contained in S (and such an element exists) then there exist elements $e(i)$ such that $pe(i) \equiv e(i-1) \bmod S$ for $i = 1, 2, \cdots$, since U/S is p-complete. The subgroup T of J which is generated by S and the elements $e(i)$ clearly satisfies (together with S and the elements $e(i)$) the conditions (a) – (d).

(5;2) *Suppose that the group J is of finite rank and not almost p-infinite. Then to every greatest linearly independent subset G of J there exists an integer w such that an equation $p^{w+k}x = \sum_{g \text{ in } G} c(g)g$ with integer coefficients $c(g)$, $0 < k$, (x in J) implies that $c(g) \equiv 0 \bmod p^k$ for every g in G.*

PROOF: G is finite, since J is of finite rank. Furthermore it is sufficient to prove the statement for $k = 1$. If this special case of (5;2) would not be true, then there would exist to every positive integer i an element $x(i)$ in J and integers $w(g, i)$ not all of them $\equiv 0 \bmod p$ such that

$$p^i x(i) = \sum_{g \text{ in } G} w(g, i)g.$$

Since G is finite, there exists an element h in G such that $w(h, i) \not\equiv 0 \bmod p$ for an infinity of numbers i. Since for these values of i the prime number p and $w(h, i)$ are relatively prime, it follows easily that there exist elements $y(i)$ in J and integers $v(g, i)$ such that

$$p^i y(i) = h + \sum_{\substack{g \text{ in } G \\ g \neq h}} v(g, i)g.$$

If H is the smallest closed subgroup of J which contains the elements $g \neq h$ in G, then $H < J$ and $p^i y(i) \equiv h \bmod H$, i.e. J is almost p-infinite and this is impossible.

THEOREM 5;3: *Suppose that g is a prime number, that F is a group without elements of infinite order, that J^* is a group without elements of finite order and that*
(1) *the orders of the elements of $C(g, F)/g^\omega C(g, F)$ are not bounded,*
(2) *J^* is almost g-infinite.*

Then there exists a group A such that $F(A) = F$, $A/F(A) = J^*$ and such that $C(g, F(A))$ is not a direct summand of A.

Since $C(g, F(A))$ is a direct summand of $F(A)$, this implies that $F(A)$ is not a direct summand of A.

PROOF: Because of (2) and (5;1) there exists a subgroup T^* of J^*, a subgroup S^* of T^* and elements $e^*(i)$ in T^* such that

S^* is closed in T^* and of finite rank,
S^* is not almost g-infinite,
$e^*(0)$ is not contained in S^*,
$s^*(i) = ge^*(i) - e^*(i-1)$ is contained in S^* (for $0 < i$),
S^* and the elements $e^*(i)$ generate T^*.

Let S be a group isomorphic with S^* and let always s and s^* correspond under this isomorphism.

By (1) and (2;1) there exist in $C(g, F)$ elements $a(i)$ such that the elements $b(i) = \sum_{j=0}^{i-1} g^j a(j+1)$ are elements of lowest order in their class mod $g^i C(g, F)$ but such that their orders tend to infinity with i.

Now let T be a group containing the direct sum $F + S$ and elements $e(i)$ such that

$$ge(i) - e(i-1) = s(i) + a(i)$$

for every positive i.

This (Abelian) group T satisfies: $F(T) = F$ and $T/F(T) = T^*$.

Suppose now that $C(g, F(T)) = C(g, F)$ is a direct summand of T. Then there exists a subgroup H of T such that $T = H + C(g, F)$. Let Q be the direct sum of all the primary-components $C(p, F)$ with $p \neq g$. Then there exists to every element x in $Q + S$ an element $f(x)$ in $C(g, F)$ and to every element $e(i)$ an element $f(i)$ in $C(g, F)$ such that $x + f(x)$ and $e(i) + f(i)$ are contained in H and these elements f in $C(g, F)$ are uniquely determined by this condition. Since H and $Q + S$ are subgroups of T, it follows therefore that

$$f(x + y) = f(x) + f(y) \text{ for } x \text{ and } y \text{ in } Q + S.$$

If x is an element in Q, the orders of x and of $f(x)$ are relatively prime and the equality for the function $f(z)$ implies therefore that $f(x) = 0$, if x is contained in Q.

(*) *The orders of the elements $f(x)$ for x in $Q + S$ are bounded.*

In order to prove this statement consider a greatest linearly independent sub-

set G of S. There exists an integer w by (5;2) such that $g^{w+k}x = \sum_{h \text{ in } g} c(h)h$ for x in S and integers $c(h), 0 < k$, implies that all the $c(h)$ are divisible by g^k.

If now x is any element in $Q + S$, then $x = y + s$ for y in Q and s in S and therefore as noted before: $f(x) = f(s)$. If $s \neq 0$, then there exists an integer v and integers $v(h)$ such that $vs = \sum_{h \text{ in } G} v(h)h$ and v and the numbers $v(h)$ have no common divisor. But this implies that v is not divisible by g^{w+1} and since G is finite and $vf(s) = \sum_{h \text{ in } G} v(h)f(h)$, we have proved (*).

Since furthermore

$$\begin{aligned} g(e(i) + f(i)) &= e(i-1) + s(i) + a(i) + gf(i) \\ &= e(i-1) + f(i-1) + s(i) + f(s(i)), \end{aligned}$$

we have:

$$(**)\quad gf(i) - f(i-1) = f(s(i)) - a(i).$$

By (*) there exists a number v such that $g^v f(x) = 0$ for x in $Q + S$. Then it follows from (**) that

$$\begin{aligned} g^{v+i}f(v+i) - f(0) &= \sum_{j=0}^{v+i-1} g^j(f(s(j+1)) - a(j+1)) \\ &= \sum_{j=0}^{v-1} g^j f(s(j+1)) - \sum_{j=0}^{v+i-1} g^j a(j+1) \end{aligned}$$

since the elements $s(j)$ are contained in $Q + S$. But this equation implies

$$f(0) + \textstyle\sum_{j=0}^{v-1} g^j f(s(j+1)) \equiv b(v+i-1) \bmod g^{v+i}C(g, F)$$

and this congruence contradicts the choice of the elements $a(i)$, since its left side does not depend on i. Hence $C(g, F(T))$ is not a direct summand of T.

By (4;3) T is contained in a group A such that $F(A) = F(T) = F$ and $A/F(A) = J$. Then $C(g, F(A))$ is not a direct summand of A, since otherwise $C(g, F(A)) = C(g, F(T))$ would be a direct summand of T and this is impossible as proved before. This completes the proof of our Theorem.

6. The second necessary condition.

(6;1) *Suppose that P is an infinite set of prime numbers and that the group J without elements of finite order is almost P-infinite. Then there exists an infinite subset P^* of P, a pair of subgroups S and T of J and to every prime number p in P^* an element $e(p)$ such that*

(a) *S is a closed subgroup of finite rank of T,*
(b) *S is not almost P-infinite,*
(c) *T is generated by the elements $e(p)$ and by the elements in S,*
(d) *$pe(p) \equiv qe(q) \not\equiv 0 \bmod S$ for every pair p, q of prime numbers in P^*.*

PROOF: Since the group J is almost P-infinite, there exists a closed subgroup U of finite rank of J, an infinite subset P' of P and elements $v(p)$ for every p in P'

such that $pv(p) \equiv qv(q) \not\equiv 0 \bmod V$ for every pair p, q of prime numbers in P'. The group V, generated by U and the elements $v(p)$, is an almost P-infinite subgroup of finite rank of J and therefore J contains an almost P-infinite subgroup W of lowest rank. W contains a closed subgroup S such that W/S is P-infinite (and of rank 1). Since the rank of S is lower than the rank of W, S is not almost P-infinite. Since W is almost P-infinite, there exists an infinite subset P^* of P and elements $e(p)$ for p in P^* such that $e(p)$ is contained in W and satifies: $pe(p) \equiv qe(q) \not\equiv 0 \bmod S$ for every pair p, q of prime numbers in P^*. S, P^*, the elements $e(p)$ and the subgroup T, generated by S and the elements $e(p)$, satisfy (a) – (d).

(6;2) *Suppose that the group J without elements of finite order is not almost P-infinite, that J is of finite rank and that G is a greatest linearly independent subset of J. Then G is independent* mod pJ *for almost every prime number p in P.*

Proof: Let W be the set of all the prime numbers p in P such that G is not independent mod pJ. Then to every prime number p in W there exists an element $w(p)$ in J and integers $w(p, g)$ such that

$$pw(p) = \sum_{g \text{ in } G} w(p, g)g,\ 0 \leqq w(p, g) < p, \text{ not every } w(p, g) = 0.$$

If W is infinite, then there exists, since G is finite, an element h in G and an infinite subset H of W such that $w(p, h) \neq 0$ for every p in H. Since $w(p, h)$ and p are relatively prime, we can assume without loss of generality that $w(p, h) = 1$ for every p in H. If V is the smallest closed subgroup of J generated by the elements $\neq h$ in G, then

$$pw(p) \equiv h \not\equiv 0 \bmod V,$$

i.e. J would be almost H-infinite and therefore almost P-infinite and this is impossible. Hence W is finite and this proves (6;2).

Theorem 6;3: *Suppose that F is a group without elements of finite order that the set P of all the prime numbers p with $pC(p, F) \neq C(p, F)$ is infinite and that the group J^* without elements of finite order is almost P-infinite. Then there exists a group A such that $F(A) = F$, $A/F(A) = J^*$ and such that $F(A)$ is not a direct summand of A.*

Proof: By (6;1) there exists an infinite subset W of P and a pair S^*, T^* of subgroups of J^* and elements $e^*(p)$ for p in W such that

S^* is a closed subgroup of finite rank of T^*,
S^* is not almost P-infinite,
T^* is generated by the elements $e^*(p)$ and by the elements of S^*,
$pe^*(p) \equiv qe^*(q) \not\equiv 0 \bmod S^*$ for every pair p, q of prime numbers in W.

Then there exists an element e^* in T^* such that

$$pe^*(p) - e^* = s^*(p) \text{ is an element of } S^*.$$

Furthermore there exists to every prime number p in W an element $f(p)$ in $C(p, F)$ which is not contained in $pC(p, F)$.

Let S be a group isomorphic with S^* and let always s and s^* correspond under this isomorphism.

T is the group containing the direct sum $F + S$ and elements $e(p)$, e for p in W such that

$$pe(p) - e = s(p) + f(p).$$

This (Abelian) group T satisfies: $F(T) = F$ and $T/F(T) = T^*$.

Suppose now that $F(T)$ is a direct summand of T. Then $T = F(T) + H$ for a suitable subgroup H of T and therefore there exists to every element x in S an element $g(x)$ in F and to every p in W an element $g(p)$ in F and an element g in F such that $x + g(x)$ and $e(p) + g(p)$, $e + g$ are contained in H. Since S and H are subgroups of T it follows that

$$g(x + y) = g(x) + g(y) \text{ for } x \text{ and } y \text{ in } S.$$

(*) *For almost every prime number p in W the relation $c(p, g(x)) = 0$ holds for every element x in S.*

In order to prove this statement consider a greatest linearly independent subset G of S. Since G is finite, there exists only a finite number of prime numbers p such that

$$c(p, g(h)) \not\equiv 0 \text{ for at least one element } h \text{ in } G.$$

Therefore it follows from (6;2) that the set V of all the prime numbers p in W satisfying

$$c(p, g(h)) = 0 \text{ for every element } h \text{ of } G$$

and

$$G \text{ is linearly independent mod } pS$$

contains almost every prime number of W.

Let now s be any element in S and p a prime number contained in V. Then there exist integers n, $d(h)$ for h in G such that

$$ns = \sum_{h \text{ in } G} d(h)h$$

and since G is independent mod pS, it can be assumed without loss of generality that n and p are relatively prime. But this relation implies

$$nc(p, g(s)) = c(p, ng(s)) = c(p, g(ns)) = \sum_{h \text{ in } G} d(h)c(p, g(h)) = 0$$

and therefore
$c(p, g(s)) = 0$ for every p in V. This proves (*), since V contains almost every element of W.

An obvious consequence of (*) is

(**) *For almost every prime number p in W the relations*

$$c(p, g(x)) = 0 \text{ for every } x \text{ in } S \text{ and } c(p, g) = 0 \text{ hold.}$$

Since $T = F(T) + H = F + H$, it follows that

$$\begin{aligned} p(e(p) + g(p)) - (e + g) &= s(p) + g(s(p)) \\ &= s(p) + f(p) + pg(p) - g \end{aligned}$$

or

$$f(p) + pg(p) = g(s(p)) + g$$

or

$$c(p, f(p)) + pc(p, g(p)) = f(p) + pc(p, g(p)) = c(p, g(s(p))) + c(p, g).$$

By (**) this implies for almost every p in W:

$$f(p) + pc(p, g(p)) = 0$$

and this is impossible, since $f(p)$ is contained in $C(p, F)$ but not in $pC(p, F)$. Hence $F = F(T)$ is not a direct summand of T.

The Theorem 6;3 is now an obvious consequence from the statement (4;4).

7. Sufficient conditions.

(7;1) *Suppose that the subgroup B of the group A satisfies the following conditions:*

(a) *A/B is a group of rank 1 without elements of finite order;*[6]

(b) *If the orders of the elements of $C(p, F(B))/p^{\omega}C(p, F(B))$ are not bounded, then A/B is not p-complete;*

(c) *If the set P of all the prime numbers p such that $C(p, F(B)) \neq pC(p, F(B))$ is infinite, then A/B is not P-infinite.*

Then to every subgroup B' of B such that $B = F(B) + B'$ there exists[7] *a subgroup A' of A such that*

$$B' \leqq A' \text{ and } A = F(A) + A'.$$

PROOF: There exists a greatest subgroup C of $F(B)$ such that $pC = C$ for every prime number p. Furthermore there exists a greatest subgroup H of A such that 0 is the only element contained in both the subgroups C and H, and it is possible to choose H in such a way that H contains B'. Then as proved in (1;1) it follows that $A = C + H$. If K is the intersection of B and H, then $B = C + K$, and the pair H, K satisfies the conditions (a) – (c) exactly as the pair A, B and therefore we can assume without loss of generality:

[6] This condition implies that $F(A) = F(B)$ and that $B/F(B)$ is a closed subgroup of $A/F(A)$.

[7] This statement is of course vacuous if $F(B)$ is not a direct summand of the group B.

(d) *If C is any subgroup of $F(B)$ such that $pC = C$ for every prime number p, then $C = 0$.*

Let now $x^* \neq 0$ be any element of the group $R^* = A/B$. Condition (c) implies that there exists only a finite number of prime numbers p in P such that x^* is the p-fold of an element in R^* and therefore there exists an element $e^* \neq 0$ in R^* such that:

for every prime number p in P either R^* is p-complete or e^* is not a p-fold.

There are three classes of prime numbers to distinguish:

The finite set U of the prime numbers u in P such that R^* is u-complete.

The set V of prime numbers v which are not contained in P such that R^* is v-complete.

The set W of prime numbers w which are not contained in P such that R^* is not w-complete.

Note that some of these classes may be vacuous and that the prime numbers of P which are not contained in U are not attributed to any one of these classes.

If r is a prime number either in U or in V, then there exist (uniquely determined) elements $e^*(r, i)$ in R^* such that

$$e^* = e^*(r, 0),\ re^*(r, i+1) = e^*(r, i).$$

If w is a prime number in W, then there exists a (uniquely determined) element $e^*(w)$ and a number $h(w)$ such that

$$e^*(w) \text{ is not a } w\text{-fold of an element in } R^* \text{ and satisfies } w^{h(w)}\, e^*(w) = e^*.$$

The elements e^*, $e^*(w)$ and $e^*(r, i)$ generate the group R^*.

Now let e, $e(w)$ and $e(r, i)$ be elements contained in the classes e^*, $e^*(w)$ and $e^*(r, i)$ of A/B respectively, and since $e^* = e^*(r, 0)$ we choose $e = e(r, 0)$. Then

$$w^{h(w)}\, e(w) - e = f(w) + b(w),$$

$$re(r, i+i) - e(r, i) = f(r, i) + b(r, i),$$

where $f(w), f(r, i)$ are uniquely determined elements in $F(B)$, $b(w)$, $b(r, i)$ uniquely determined elements in B', (since $B = F(B) + B'$).

There exists a subgroup A' of A which contains B' and satisfies $A = F(A) + A'$ if, and only if, there exist solutions g, $g(w)$, $g(r, i)$ of the equations

$$w^{h(w)}\, (e(w) + g(w)) - (e + g) = b(w),$$

$$r(e(r, i+1) + g(r, i+1)) - (e(r, i) + g(r, i)) = b(r, i),\ g(r, 0) = g,$$

in $F(B)$. Equivalent with these equations are the equations:

$$w^{h(w)}\, g(w) - g = -f(w),$$

$$rg(r, i+1) - g(r, i) = -f(r, i),\ g(r, 0) = g.$$

If the prime number q is not contained in P, i.e. if q is contained in V or W, then $C(q, F(B)) = qC(q, F(B))$ and therefore by condition (d): $C(q, F(B)) = 0$,

i.e. the orders of all the elements in $F(B)$ are relatively prime to q. Therefore there exist to every given element g uniquely determined solutions $g(w)$ and $g(v, i)$ in $F(B)$.

Similarly it follows from the finiteness of U that it is sufficient to solve the equations $uh(u, i+1) - h(u, i) = -c(u, f(u, i))$, $h(u, 0) = h$ in $C(u, F(B))$.

From (b), (d) and (2;1) it follows that the orders of the elements in $C(u, F(B))$ are bounded, i.e. that there exists a number $m = m(u)$ such that $u^m x = 0$ for every element x in $C(u, F(B))$. But then the elements

$$h(u, i) = \sum_{j=0}^{m-1} u^j c(u, f(u, i+j))$$

are contained in $C(u, F(B))$ and satisfy:

$$uh(u, i+1) - h(u, i) = -c(u, f(u, i))$$

and this completes the proof of (7;1).

REMARK: By an argument similar to that used in proving the Theorems 5;3 and 6;3 it can be shown that the conditions (b) and (c) are also necessary for the existence of the direct summands A' containing B' for every possible B'.

(7;2) *Suppose that the group A satisfies the following conditions:*

(a) *$A/F(A)$ is countable;*

(b) *If the orders of the elements of $C(p, F(A))/p^\omega C(p, F(A))$ are not bounded, then $A/F(A)$ is not almost p-infinite;*

(c) *If the set P of all the prime numbers p such that $C(p, F(A)) \neq pC(p, F(A))$ is infinite, then $A/F(A)$ is not almost P-infinite.*

Then there exists to every subgroup B of A which exactly represents[8] a closed subgroup of finite rank of $A/F(A)$ a subgroup A' of A such that

$$B \leqq A' \text{ and } A = F(A) + A'.$$

PROOF: Let B^* be the subgroup of $A^* = A/F(A)$ which is represented by the elements in B. Then there exists because of condition (a) a sequence $B^*(i)$ of subgroups of A^* such that

$$B^* = B^*(0),$$

$B^*(i)$ is a closed subgroup of finite rank of A^*,

$$B^*(i) < B^*(i+1),$$

$$B^*(i+1)/B^*(i) \text{ is a group of rank 1,}$$

either $A^* = B^*(m)$ for a certain integer m or A^* is the join of all the $B^*(i)$.

Assume that there exist groups $A(i)$ for $0 \leqq i \leqq k$ such that $B = A(0)$, $A(i) < A(i+1)$, $A(i)$ represents exactly $B^*(i)$. Let $B(i)$ be the subgroup of A which contains $F(A)$ and satisfies $B(i)/F(A) = B^*(i)$. Then $B(k) = F(A) + A(k)$ and it follows from (b), (c) and (7;1) applied on $B(k)$, $B(k+1)$ and $A(k)$ that there

[8] I.e. B contains at most one element of a class of $A/F(A)$.

exists a subgroup $A(k+1)$ of $B(k+1)$ such that $A(k) < A(k+1)$ and $B(k+1) = F(A) + A(k+1)$. Hence these subgroups $A(i)$ exist for every i. If A' is the join of all the $A(i)$, then

$$B \leqq A' \text{ and } A = F(A) + A'.$$

(7;3) *Suppose that the group A satisfies the following conditions*:

(a) *If the orders of the elements of $C(p, F(A))/p^{\omega}C(p, F(A))$ are not bounded, then $A/F(A)$ is not almost p-infinite.*

(b) *If the set P of all the prime numbers p such that $pC(p, F(A)) \neq C(p, F(A))$ is infinite, then $A/F(A)$ is not almost P-infinite.*

Suppose that the subgroup B of A satisfies the following conditions:

(c) *The intersection of B and $F(A)$ is 0.*

(d) *The subgroup B^* of $A^* = A/F(A)$ which is represented by elements of B is closed in A^* and of finite rank.*

(e) *$D(A^*/B^*)$ exists.*[9]

Then there exists a subgroup A' of A such that $B \leqq A'$ and $A = F(A) + A'$.

PROOF: If $D(A^*/B^*) = 1$, then A^* is countable, since B^* is a subgroup of finite rank. This case of (7;3) is therefore a consequence of (7;2). Hence we can prove (7;3) by complete (transfinite) induction with regard to $v = D(A^*/B^*)$.

Since $D(A^*/B^*)$ exists, there exists a closed subgroup S^{**} of $A^{**} = A^*/B^*$ of finite rank such that $A^{***} = A^{**}/S^{**}$ is the direct sum of groups $A^{***}(w)$ satisfying $D(A^{***}(w)) < D(A^{**}) = v$. Then let $A^{**}(w)$ be the subgroup of A^{**} which contains S^{**} and satisfies $A^{**}(w)/S^{**} = A^{***}(w)$, S^* the subgroup of A^* corresponding to S^{**}, $A^*(w)$ the subgroup of A^* corresponding to $A^{**}(w)$, finally $A(w)$ and S the subgroups of A which contain $F(A)$ and correspond to $A^*(w)$ and S^* respectively.

Since S^* is of finite rank (and therefore countable), there exists from (a), (b) and (7;2) a subgroup S' of S such that $B \leqq S'$ (for B is a subgroup of S) and $S = F(A) + S'$.

Since $A^*(w)/S^*$ and $A^{***}(w)$ are isomorphic, $D(A^*(w)/S^* = D(A^{***}(w)) < v$ and there exists therefore by the hypothesis of the induction a subgroup $A'(w)$ of $A(w)$ such that $S' \leqq A'(w)$ and $A(w) = F(A) + A'(w)$.

Let A' be the subgroup of A, generated by the subgroups $A'(w)$. Then every class of $A/F(A)$ is represented by elements in A', since A/S is the direct sum of the groups $A(w)/S$. Furthermore every element of A' has the form $\sum_{i=1}^{k} x(i)$ where the elements $x(i)$ belong to different groups $A'(w)$. If this element is contained in $F(A)$, then $\sum_{i=1}^{k} x(i) \equiv 0 \bmod S$, i.e. $x(i) \equiv 0 \bmod S$, since A/S is the direct sum of the groups $A(w)/S$. Since $x(i)$ is an element of S and of an $A'(w)$, $x(i)$ is an element of S' and therefore $\sum_{i=1}^{k} x(i)$ is an element of S'. But this implies that $\sum_{i=1}^{k} x(i) = 0$, since the intersection of $F(A)$ and S' is 0, i.e. the intersection of A' and $F(A)$ is 0 and therefore $A = A' + F(A)$. Furthermore $B \leqq S' \leqq A'$ and this completes the proof.

[9] As a consequence of (3;3) it follows from this condition (e) that $D(A^*)$ exists and satisfies $D(A^*) \leqq D(A^*/B^*)$.

8. **Final conclusions.**

THEOREM 8;1: *Suppose that F is a group without elements of infinite order, that J is a group without elements of finite order and that $D(J)$ exists. Then $F(A)$ is a direct summand of every group A such that $F(A) = F$ and $A/F(A) = J$ if, and only if, the following two conditions are satisfied:*

(a) *If the orders of the elements of $C(p, F)/p^{\omega}C(p, F)$ are not bounded, then J is not almost p-infinite.*

(b) *If the set P of all the prime numbers p such that $C(p, F) \neq pC(p, F)$ is infinite, then J is not almost P-infinite.*

PROOF: The necessity of condition (a) is a consequence of Theorem 5;3 and the necessity of condition (b) is a consequence of Theorem 6;3.

That the conditions are sufficient is a consequence of (7;3), if we choose as the subgroup B of A the 0-group which satisfies all the conditions.

Since the existence of $D(J)$ has not been used in the proof of the necessity of the conditions, we have even proved the

COROLLARY 8;2: *If $F(A)$ is a direct summand of every group A such that $F(A) = F$ and $A/F(A) = J$, then the groups F and J satisfy the conditions* (a) *and* (b).

COROLLARY 8;3: *Suppose that F is a group without elements of infinite order, that the group J is contained in a group J' without elements of finite order such that $D(J')$ exists and the pair F, J' satisfies the conditions* (a) *and* (b).[10] *Then $F(A)$ is a direct summand of every group A such that $F(A) = F$ and $A/F(A) = J$.*

This Corollary is a consequence of the Theorem 8;1 and of (4;4).

COROLLARY 8;4: *Suppose that F is a group without elements of infinite order, that J is a group without elements of finite order and that $D(J)$ exists. Then $C(p, F(A))$ is a direct summand of every group A such that $F(A) = F$ and $A/F(A) = J$ if, and only if, the condition* (a) *is satisfied for this particular prime number p.*

This is a consequence of the Theorem 8;1 and of (4;5), since for primary groups F the condition (b) becomes void.

THEOREM 8;5: *Let F be a group without elements of infinite order. Then $F(A)$ is a direct summand of every group A such that $F(A) = F$ if, and only if,*

(1) *the orders of the elements of $C(p, F)/p^{\omega}C(p, F)$ are bounded for every prime number p;*

(2) *$C(p, F) = pC(p, F)$ for almost every prime number p.*

PROOF: The necessity of the conditions is a consequence of the Corollary 8;2, since we can choose as a group J the additive group of all the rational numbers. That the conditions are sufficient, is a consequence of the Corollary 8;3, since by (3;1) every group J is contained in a complete group J', and since complete groups are direct sums of coutable groups, i.e. $D(J') \leqq 2$.

THEOREM 8;6: *Suppose that J is a group without elements of finite order and that $D(J)$ exists. Then $F(A)$ is a direct summand of every group A such that $A/F(A) = J$ if, and only if,*

[10] This implies that the pair of groups F, J satisfies also the conditions (a) and (b).

for every closed subgroup S of J of finite rank all the (closed) subgroups of rank 14 *of J/S are cyclic groups.*[11]

PROOF: It is easily inferred from Theorem 8;1 that $F(A)$ is a direct summand of every group A such that $A/F(A) = J$ if, and only if, J is neither almost p-infinite for any prime number p nor almost P-infinite for any infinite set P of prime numbers. But this condition is clearly equivalent with the condition of the Theorem.

Note that by Corollary 8;2 the condition of the Theorem is also necessary if $D(J)$ does not exist.

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[11] It can be proved that this condition together with the existence of $D(J)$ imply that J is a direct sum of cyclic groups. But there exist groups which satisfy this condition and which are not direct sums of cyclic groups (e.g. the additive group of all the sequences of integers) and for these groups the problem whether the condition of the Theorem 8;6 is a sufficient one is still unsettled.

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ON EQUIVALENT SETS OF ELEMENTS IN A FREE GROUP

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1. Let G be the free group generated by the symbols $a_1, \cdots, a_p$ and their inverses[1] $a_1', \cdots, a_p'$, and let $(\alpha) = (\alpha_1, \cdots, \alpha_m)$ be a set of elements of G which generate a sub-group Γ. In this paper we consider two kinds of equivalence between sets (α) and (β). The first is equivalence under the automorphisms of Γ, generated by 'simple automorphisms' of the form

$$(1.1) \quad \alpha_\lambda^* = \alpha_\lambda \alpha_\rho^\epsilon \text{ or } \alpha_\rho^\epsilon \alpha_\lambda, \qquad \alpha_\mu^* = \alpha_\mu \qquad (\rho \neq \lambda, \quad \mu \neq \lambda, \quad \epsilon = \pm 1),$$

together with the 'simple automorphism' which replaces α_1 by its inverse. Relative to this kind of equivalence we have very little to add to a paper by J. Nielsen,[2] in which he gives a mechanical process for deciding whether or no two sets are equivalent and also a process for reducing (α) to one of a finite number of normal forms. When reduced in this way we shall describe a set of elements as *reduced* (N), and we recall that (α) is reduced (N) if it contains no two words of the form $(AB)^{\epsilon_1}$ and $(AC)^{\epsilon_2}$ respectively,[3] where $l(A) > l(B)$ or $> l(C)$, and if the last half of every word with an even number of letters is an 'isolated ending.' That is to say, if $\alpha \equiv AB$ and $l(A) = l(B)$ no other word in (α) ends with B or begins with B'.

In §2 it is assumed that any empty words which appear[4] during the process of reduction are discarded, as in J. N., while in §4 they are retained.

Theorem 1 below is essentially a restatement of various arguments used by Nielsen, while Theorem 2 adds a detail to J. N.

The second kind of equivalence refers to the effect on (α) of automorphisms[5] of G, two ordered sets of elements (α) and (β), both of which contain the same number of words, being equivalent if α_λ corresponds to $\beta_\lambda (\lambda = 1, 2, \cdots)$ in some automorphism of G. That is to say they are equivalent if there is an automorphism

$$a_i \to f_i(a, a') = f_i(a_1, \cdots, a_p, a_1', \cdots, a_p'),$$

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[1] As in a recent paper (J. H. C. Whitehead, *Proc. L. M. S.*, 41 (1936), 48–56, which will be referred to as W. G., W' will be used as an abbreviation for W^{-1}.

[2] *Mat. Tidsskrift*, (1921, B), 77–94. This paper will be referred to as J. N.

[3] $l(W)$ stands for the number of letters in W. Here we assume that each word in (α) is reduced (E) (see the conclusion of this section).

[4] This can happen after a transformation of the form (1·1), where $\alpha_\rho^\epsilon = \alpha_\lambda'$ but $\rho \neq \lambda$.

[5] For a far reaching account of the group of automorphisms of G see Nielsen, *Math. Annalen*, 91 (1924), 169–209.

such that

$$\beta_\lambda(a, a') = \alpha_\lambda[f(a, a'), f'(a, a')].$$

Nielsen[6] has shown that the automorphisms of G are generated by simple automorphisms of the form $a_i^\epsilon \to a_i^\epsilon a_j^{\epsilon_1}$, $a_k \to a_k$ $(j, k \neq i)$, together with $a_1 \to a_1'$. Thus the problem is to determine whether or no an ordered set (α) can be transformed into an ordered set (β) by a sequence of what I have called *simple transformations*,[7] among which we now include $a_1 \to a_1'$. In §3 this problem is solved for finite sets. We adopt the convention followed in W. G. relative to true and cyclic words, a cyclic word being a sequence of letters in cyclic order, and in §3 any statement about a set of words will apply equally well to a set of true and to a set of cyclic words. Thus the equivalence problem is solved both for ordered sets of elements and for ordered sets of classes of elements.

We shall follow H. Seifert and W. Threlfall[8] in using the symbol $\equiv$ to denote equality between words and $=$ for equality between elements. Thus the true word $abb' = a$ but $\not\equiv a$ and the cyclic word $bab' = a$. If $W(a, a')$ contains no syllable xx' we shall describe it as 'reduced (E)' or as 'reduced (E, a)', and 'reduced (E, α)' will have a similar meaning relative to $F(\alpha, \alpha')$, where $\alpha_\lambda = \alpha_\lambda(a, a')$.

2. Let Γ be the sub-group of G generated by (α), where

$$\alpha_\lambda = \alpha_\lambda(a, a'),$$

and α_λ is reduced (E). Let

$$W(a, a') \equiv F(\alpha, \alpha') = W^0(a, a'),$$

be any element in Γ, where F is reduced (E, α) and W^0 is reduced (E, a). We represent W by an oriented polygonal segment $S(W)$, in a Euclidean plane, the oriented edges of which represent the letters of W in their correct order, a_i pointing the same way as $S(W)$ and a_i' pointing backwards. Whenever a syllable xx' is omitted during the reduction $W \to W^0$ we identify the corresponding edges of $S(W)$ in such a way that their common end point is on the left hand side of the new $S(W)$, positively described. Finally $S(W)$ will have been transformed into a figure $S^0(W)$, consisting of a segment $S^0(W^0)$ with a row of trees $T_1, \cdots, T_q$ attached to its left hand side, and in spelling out W each edge of $S^0(W^0)$ is described once and each edge of T_ρ twice, with opposite senses. The

[6] *Math. Annalen*, 79 (1919), 269–72.

[7] W. G., §1. A simple transformation of a set of words (W), consists either of replacing x^ϵ by $(xy)^\epsilon (y \neq x \text{ or } x')$ whenever x^ϵ occurs in (W), or of an elementary transformation, which consists of omitting a syllable xx' (elementary reduction) or of inserting one (elementary insertion).

[8] *Lehrbuch der Topologie*, Leipzig (1934), chap. 12.

vertex in which T_ρ meets $S^0(W^0)$ will be called its *base*, and the other extremities of T_ρ will be called its *leaves*. It is to be understood that each tree has but a single edge incident with its base. Thus two or more trees may have a common base. By the length $l(s)$, of any segment[9] s in $S^0(W)$ we shall mean the number of edges in s, and h_ρ will stand for the longest segment joining a leaf of T_ρ to its base. We assume $T_1, \cdots, T_q$ to be the order in which the trees occur in spelling out W and shall denote by l_ρ the length of the segment, possibly zero, joining the base of T_ρ to the base of $T_{\rho+1}$, T_0 and T_{q+1} being the first and last vertices of $S^0(W^0)$. Finally, we shall describe T_ρ as a *stump* if it has but a single leaf.

THEOREM 1. *If (α) is reduced (N), then each tree in $S^0(W)$ is a stump, no word in $F(\alpha, \alpha')$ ends at a vertex of T_ρ other than a leaf, and*

$$(2.1) \qquad \begin{cases} \text{a} & l_\rho \geqq |\, h_{\rho+1} - h_\rho \,| \ (h_0 = h_{q+1} = 0), \\ \text{b} & l_\rho > 0. \end{cases}$$

This theorem, being vacuous if $F \equiv 1$, will be proved by induction on $l(F)$, measured in terms of the α's. We first show that no word in F, except possibly the last, ends at a vertex P, of T_ρ, which is not a leaf. For if it does, let

$$F(\alpha, \alpha') \equiv F_1(\alpha, \alpha')F_2(\alpha, \alpha'),$$

where F_1 ends at P. If P is the base of T_ρ let L be any leaf, and if P is not the base let L be any leaf which is separated from the base by P. Then the branch consisting of all the edges of T_ρ which are not separated from L by P is a tree belonging either to F_1 or to F_2, say to F_1. If

$$W_1(a, a') = F_1(\alpha, \alpha'),$$

the figure $S^0(W_1)$ has a tree growing out of its last vertex, contradicting the hypothesis $l_q > h_q$, calculated for $S^0(W_1)$. A similar contradiction arises if a branch above P belongs to F_2.

If $S^0(W)$ contains no trees there is nothing to prove. Otherwise let

$$F \equiv \alpha_\lambda^\epsilon F_1(\alpha, \alpha') \equiv \alpha_\lambda^\epsilon W_1(a, a'),$$

where

$$\alpha_\lambda^\epsilon \equiv AB, \qquad W_1 \equiv B'U(a, a'),$$

and the last letter in A is not the inverse of the first letter in U. From the preceding paragraph it follows that the last vertex of α_λ^ϵ either precedes the base of T_1 or is a leaf of T_1. In the first case the theorem follows immediately from the hypothesis of the induction. In the second case let α_μ^ϵ be the second word in F. From the hypothesis of the induction it follows that every tree in $S^0(W_1)$ is a stump, and if T_1 were not a stump α_μ^ϵ would be completely absorbed by B and the beginning of the third word in F, contradicting the assumption that

[9] By a segment we shall always mean a non-singular segment.

(α) is reduced (N). Therefore T_1 is a stump, α_μ^ϵ is of the form $B'C$, where $l(B) \leqq l(C)$, and T_1 spells out B, read from base to leaf. Since (α) is reduced (N) we have

$$l_0 = l(A) \geqq l(B) = h_1,$$

$$l_1 + h_2 \geqq l(C) \geqq h_1,$$

and the remaining inequalities (2.1a) follow from the hypothesis of the induction, applied to $S^0(W_1)$

If $l_\rho = 0$ the word starting at the leaf of T_ρ would finish at the leaf of $T_{\rho+1}$ and would be absorbed into its two neighbours, contradicting the assumption that (α) is reduced (N). Therefore $l_\rho > 0$, and the theorem is established.

Two important results[10] are obvious consequences of this theorem, namely:

If the set (α) *is reduced* (N) *it is independent,*[11] and:

If (α) *is reduced* (N) *and* $\Gamma = G$ *each* α *is an* a^ϵ.

It also follows[12] that $l(W^0) \geqq l(\xi)$, where ξ is any α^ϵ in

$$F(\alpha, \alpha') = W^0(a, a').$$

For let the segment representing ξ occur after the leaf of T_ρ and before the leaf of $T_{\rho+1}$. Then $l(\xi) \leqq h_\rho + l_\rho + h_{\rho+1}$, and from (2.1a) we have

$$l_0 + \cdots + l_{\rho-1} \geqq h_\rho, \qquad h_{\rho+1} \leqq l_{\rho+1} + \cdots + l_q.$$

Therefore

$$l(W^0) = l_0 + \cdots + l_q \geqq h_\rho + l_\rho + h_{\rho+1} \geqq l(\xi).$$

If $l(W^0) = l(\xi)$, ξ being one of the longest words in F, it follows that $l(\xi) = h_\rho + l_\rho + h_{\rho+1}$ and that $h_\sigma + l_\sigma = h_{\sigma+1}$ if $\sigma < \rho$, $h_\sigma = l_\sigma + h_{\sigma+1}$ if $\sigma > \rho$. Let this be the case and let $\xi_1, \cdots, \xi_s$ be the longest words in F, all of equal length, and let $\eta_1, \cdots, \eta_t$ be the remaining words, if there are any, each being shorter than the ξ's. I say that

$$F = F_1(\eta, \eta')F_0(\xi, \xi')F_2(\eta, \eta'),$$

where $F_0 = \xi_\rho^\epsilon$ or $\xi_\rho' \xi_\sigma$. For if F begins and ends with a ξ^ϵ the assertion follows from (2.1b) and the fact that

$$l_0 \geqq \tfrac{1}{2} l(\xi), \qquad l_q \geqq \tfrac{1}{2} l(\xi)$$

if $q > 0$, while $\epsilon_1 = -1$ and $\epsilon_2 = 1$ if

$$F \equiv \xi_\rho^{\epsilon_1} \xi_\sigma^{\epsilon_2},$$

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[10] J. N., theorem 1, and *Math. Annalen*, 79 (1919), 269–72.

[11] This means that $W(a, a') \neq 1$, where $W(a, a') \equiv F(\alpha, \alpha')$, if $F \neq 1$ and F is reduced (E, α).

[12] J. N., p. 91.

since the last half of each ξ is an isolated ending. If F begins or ends with an η^ϵ we have $l_0 = h_1$ or $l_q = h_q$ and, removing the extreme η^ϵ, the assertion follows from induction on the number of words in F. It also follows that each η^ϵ in F_1 is an η' and each η^ϵ in F_2 is an η, since the last half of any η with an even number of letters is an isolated ending.

By a *level* transformation we shall mean either a transformation of the form $(\alpha) \to (\alpha^*)$, where (α^*) is any independent set of elements among $\alpha_1, \cdots, \alpha_m, \alpha_1', \cdots, \alpha_m'$, or a transformation of the form (1.1) such that $l(\alpha_\lambda^*) = l(\alpha_\lambda)$. Thus $\alpha_\lambda^* = \alpha_\lambda \alpha_\rho$ is a level transformation if, and only if, $\alpha_\lambda \equiv AB$, $\alpha_\rho \equiv B'C$, where $l(B) = l(C)$ and AC is reduced (E, a). Notice that the inverse of a level transformation is a level transformation. We shall describe (α) as a *minimal* set if

$$L(\alpha) = \sum_{\lambda=1}^{m} l(\alpha_\lambda)$$

is a minimum among all sets of generators for Γ. From the rule given by Nielsen for reduction (N) it is clear that the reduction of any minimal set to one which is reduced (N) consists of a sequence of level transformations, and from J. N., theorem 6, it follows that any reduced system (N) is minimal.

THEOREM 2. *Any two minimal sets of generators for* Γ *are interchangeable by level transformations.*

Let (α) and (β) be any two minimal sets of generators for Γ. From the preceding paragraph it follows that no generality is lost in assuming (α) and (β) to be reduced (N). Then they both contain the same number of elements, say m, and, assuming them to be ordered so that

$$l(\alpha_\lambda) \leqq l(\alpha_\mu), \qquad l(\beta_\lambda) \leqq l(\beta_\mu)$$

if $\lambda < \mu$, we have [13]

$$l(\alpha_\lambda) = l(\beta_\lambda) = k_\lambda \qquad (\lambda = 1, \cdots, m).$$

If $m = 0$ the theorem is vacuous, and we shall prove it by induction on m. Let k_s be the first of $k_1, \cdots, k_m$ which is the same as k_m. If $s > 1$ and $\lambda < s \leqq \rho$ we have $l(\beta_\lambda) < l(\alpha_\rho)$, $l(\alpha_\lambda) < l(\beta_\rho)$ and it follows from a previous result that $(\beta_1, \cdots, \beta_{s-1})$ depend only on $(\alpha_1, \cdots, \alpha_{s-1})$ and vice versa. Any sub-set of a minimal set is obviously minimal and if $s > 1$ it follows from the hypothesis of the induction that $(\beta_1, \cdots, \beta_{s-1}) \to (\alpha_1, \cdots, \alpha_{s-1})$ by a sequence of level transformations. Moreover, it follows from a previous argument that

$$(\alpha_1, \cdots, \alpha_{s-1}, \beta_s, \cdots, \beta_m) \to (\alpha_1, \cdots, \alpha_{s-1}, \gamma_s, \cdots, \gamma_m)$$

by level transformations, where

$$\gamma_\lambda = \alpha_{\rho_\lambda}^\epsilon \text{ or } \alpha_{\rho_\lambda}' \alpha_{\sigma_\lambda} \qquad (s \leqq \lambda, \rho_\lambda, \sigma_\lambda \leqq m).$$

[13] J. N., p. 91.

If $s = 1$ it follows from the same argument that $(\beta) = (\gamma)$ is related to (α) in this way.

If each γ is an α^ϵ a further level transformation $(\gamma) \to (\alpha_s, \cdots, \alpha_m)$ carries $(\alpha_1, \cdots, \alpha_{s-1}, \gamma_s, \cdots, \gamma_m)$ into (α). Otherwise we can arrange the notation so that

$$\gamma_\lambda = \alpha_\lambda \qquad (\lambda = s, \cdots, t)$$

$$\gamma_\mu = \alpha'_{\rho_\mu}\alpha_{\sigma_\mu} \qquad (\mu = t+1, \cdots, m),$$

with the convention $t = s - 1$ if no γ is an α^ϵ. If $\rho_\mu > t$, $\sigma_\mu > t$ for every value of μ the determinant of the linear transformation

$$\gamma_\lambda = \alpha_\lambda$$

$$\gamma_\mu = \alpha_{\sigma_\mu} - \alpha_{\rho_\mu} = -\alpha_{\rho_\mu} + \alpha_{\sigma_\mu},$$

calculated to the modulus 2, would be zero. Therefore some γ is an α, and $\rho_\mu \leqq t$ or $\sigma_\mu \leqq t$ for at least one value of μ, say $\gamma_{t+1} = \alpha'_t\alpha_\sigma$. Then $\sigma > t$, or the set (γ) would not be independent, and we may suppose $\sigma = t + 1$. Applying the level transformation

$$\gamma^*_{t+1} = \gamma_t\gamma_{t+1} = \alpha_{t+1}$$

to (γ), the theorem follows from a second induction on $m - t$.

It is a straightforward matter to write down all the sets obtainable from a given minimal set (α^0), by sequences of level transformations. For let $(\alpha^1), \cdots, (\alpha^{k_1})$ be all the sets obtained from (α^0) by single level transformations. Omitting those which already appear in the sequence, let $(\alpha^{k_1+1}), \cdots, (\alpha^{k_2})$ be the totality of sets obtained from each of $(\alpha^1), \cdots, (\alpha^{k_1})$ by single level transformations, and let $(\alpha^{k_s+1}), \cdots, (\alpha^{k_{s+1}})$ be similarly obtained from $(\alpha^{k_{s-1}+1}), \cdots, (\alpha^{k_s})$ $(s = 1, 2, \cdots)$. This process will come to an end after a number of steps which is bounded above by $(2p)^{L(\alpha^0)}$, since this number exceeds the number of sets (ξ), for which $L(\xi) = L(\alpha^0)$. If (ξ) is any set derived from (α^0) by a sequence of level transformations let $(\xi^{\rho+1})$ be derived from (ξ^ρ) by a level transformation, where $\rho = 0, \cdots, N$, $(\xi^0) = (\alpha^0)$, $(\xi^N) = (\xi)$, and[14] $(\xi^\rho) \neq (\xi^\sigma)$ if $\rho \neq \sigma$. Then it follows from induction on ρ that (ξ^ρ) is one of $(\alpha^{k_{\rho-1}}), \cdots, (\alpha^{k_\rho})$, where $k_0 = 1$. Therefore the sequence $(\alpha^0), (\alpha^1), \cdots$ contains every set which can be obtained from (α^0) by a sequence of level transformations. Thus a reduced set (β) will be contained in this sequence if it is equivalent to (α^0).

'In general' no level transformation of the form (1.1) is applicable to a reduced set (α). In such a case we have only to compare (β, β') with the various permutations of (α, α') to discover whether or not (α) is equivalent to a given reduced set (β).

Following the analogy with linear dependence, one might enquire whether or

[14] If $(\xi^\rho) = (\xi^\sigma)$ where $\sigma > \rho$ we should simply omit $(\xi^{\rho+1}), \cdots, (\xi^\sigma)$ from the sequence.

nó it is always possible to select an independent basis for Γ from a basis between whose elements there are one or more identical relations. The example

$$\alpha = a^2ba'b', \qquad \beta = b^2ab'a', \qquad \gamma = aba'b'$$

shows that the answer is in the negative. For $a = \alpha\gamma'$, $b = \beta\gamma$, but no two of (α, β, γ) generate the free group $G(a, b)$.

3. We now consider the equivalence of ordered sets of elements under the automorphisms of G. A simple transformation, operating on a set of words (W), true or cyclic, will mean the same as in §1 of W. G., referred to in §1 above, except that we shall include $a_1 \to a_1'$ among the simple transformations. By a *transformation of type T*, or a *T-transformation*, we shall mean a transformation of the form[15]

$$(3.1) \qquad a_i^{\epsilon}{}_{\rho}^{\rho} \to a_i^{\epsilon}{}_{\rho}^{\rho} a_k^{\epsilon} \qquad (k \text{ fixed}, \ \rho = 1, \cdots, r; \ i_\rho \neq k),$$

followed by what we shall call a *complete elementary reduction*, meaning a transformation by elementary reductions to words which are reduced (E). Here it is to be understood that $a_i^{\epsilon}{}_{\rho}^{\rho} \neq a_i^{\epsilon}{}_{\sigma}^{\sigma}$ if $\rho \neq \sigma$, but we allow $i_\rho = i_\sigma$, $\epsilon_\rho = -\epsilon_\sigma$, in which case

$$a_{i_\rho} \to a_k^{-\epsilon} a_{i_\rho} a_k^{\epsilon}.$$

If (W) is reduced (E) and if (W^*) is obtained from (W) by a T-transformation we shall describe the latter as a *reduction* (T), or as a *T-reduction*, if

$$L(W^*) < L(W),$$

and as a *level transformation* if

$$L(W^*) = L(W).$$

We shall also describe an automorphism of the form[16]

$$a_i \to a_j^{\epsilon}{}_i^i$$

as a level transformation. A set to which no T-reduction is applicable will be described as *minimal*, or as minimal (T) if confusion with the minimal sets of §2 is to be feared. The latter may then be described as minimal (N). The total number of T-transformations is finite and one can find out if a given one, applied to a given set, is a reduction or a level transformation or neither. Also the sets (W^*), such that $(W) \to (W^*)$ by level transformations, can be written down by a process similar to the one described above.

We shall also introduce a new use of the term elementary transformation. By an *elementary transformation* we shall now mean any one of the following:

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[15] Cf. W.G., §2.

[16] Such a transformation is a product of simple transformations (Nielsen, *Math. Annalen*, 79 (1919), 269–72).

1. The identical transformation,
2. An elementary reduction or insertion, these being defined as usual,
3. A transformation of the form[17]

$$W_\lambda \to x' W_\lambda x \qquad (\lambda = 1, \cdots, m),$$

where W_λ may be true words provided x is the same for each value of λ,

4. The resultant of any sequence of these.

It appears that the equivalence test is analogous to the one described at the end of §2. If we reduce (W) to a minimal set (W^0) by a random sequence of T-reductions, all the minimal sets which are equivalent to (W) can be obtained from (W^0) by level transformations.

To prove this we shall use the model for (W) described in W. G. §3, but will denote the 2-spheres in M by Σ_i instead of Γ_i. Let $\Sigma_1^0, \cdots, \Sigma_p^0$ be any set of oriented, non-intersecting surfaces in M, and let $w_1, \cdots, w_m$ be any set of oriented circuits[18] in M which, with the 2-spheres Σ, represent the words (W). Then the surfaces Σ^0, together with the circuits w, represent a set of words (W^0), a positive intersection $\Sigma_i^0 \cdot w_\lambda$ being marked a_i, and a negative intersection $\Sigma_i^0 \cdot w_\lambda$ being marked a_i'. Thus the substitution $\Sigma_i \to \Sigma_i^0$, which consists of replacing Σ_i by Σ_i^0 in the machine for spelling out words, determines a transformation $W_\lambda \to W_\lambda^0$.

Now let the surfaces $\Sigma_1^0, \cdots, \Sigma_p^0$ be non-singular 2-spheres, no linear combination of which bounds, and let $S_1^0, \cdots, S_q^0$ be the connected surfaces in the model R, obtained by cutting M along $\Sigma_1 \cdots, \Sigma_p$ (cf. W. G. §3), which correspond to regions on the 2-spheres Σ^0, the surfaces S^0 being bounded, in general, by circuits on the 2-spheres C_i and C_i'. Using a method due to J. Singer,[19] as adapted in W. G. (p. 54), we shall prove two lemmas, of which the first is:

LEMMA 1. *The 2-spheres Σ^0 are similar to the 2-spheres Σ in that M is transformed into a punctured sphere[20] R^0, by cutting along $\Sigma_1^0, \cdots, \Sigma_p^0$.*

To prove this we modify Singer's argument as follows. We replace his diagram Λ' by our model R, his circuits e by our 2-spheres Σ^0 and his circuits e' by our 2-spheres Σ. We replace the points P_1 and P_2 of the third paragraph on his p. 106 by a circuit belonging to some intersection $\Sigma_i^0 \cdot \Sigma$, which bounds a 2-element on Σ_i^0 containing no other circuit belonging to an intersection $\Sigma_i^0 \cdot \Sigma$. We also define moves analogous to his IA, IB and IC. A move of type IA changes the orientation of a 2-sphere Σ and determines a simple transformation of the form $a_i \to a_i'$. Let C^* be a 2-sphere in R which either

[17] On p. 50 of W. G. it is shown that such a transformation is the product of simple transformations.

[18] As in W. G., all the chains referred to will have general intersections, relative to slight deformations, unless the contrary is stated.

[19] *Trans. American Math. Soc.*, 35 (1933), pp. 105–6.

[20] By a punctured sphere is meant a bounded manifold which is equivalent to a 3-sphere from which a (finite) number of 3-elements have been removed, no two of the latter having a common vertex.

1. separates some $C_j^\epsilon (= C_j \text{ or } C_j')$ and no other 2-sphere C_i^ϵ from $C_j^{-\epsilon}$, or
2. separates $C_j^{\epsilon j}$ and precisely one other 2-sphere $C_i^{\epsilon i}$, from $C_j^{-\epsilon j}$.

We cut R along C^* and re-attach the two regions so formed by uniting corresponding vertices of C_j and C_j'. This is a move of type IB in the first case and of type IC in the second.

Let U be the region, bounded by C^* and C_j^ϵ, which is detached from R in a move of type IB, and let $C^{*\prime}$ be the side of C^* incident with U. Let U^* be the region into which U is transformed[21] by uniting C_j with C_j', let $R_1 = R - U$, $R^* = R_1 + U^*$ and let E be a 3-element bounded by $C^{*\prime}$ but having no internal component in common with R. Then R_1 and U are punctured spheres, $U^* + E$ is a 3-element, $R_1 + (U^* + E)$ is a punctured sphere and $R^* = (R_1 + U^* + E) - E$ is a punctured sphere. Thus a move of type IB transforms R into another punctured sphere. A similar argument shows that a move of type IC also transforms R into another punctured sphere, and Singer's argument shows that one can, by a series of these moves, transform R into a punctured sphere R^0, whose boundary 2-spheres correspond in pairs to the 2-spheres Σ^0. Thus lemma 1 is established.

The second lemma is:

LEMMA 2. *The set of words* (W^0) *is equivalent to the set* (W).

First let $R^0 = R$, that is to say let each Σ^0 be a Σ. Then we can transform (W) into (W^0) by a change of notation

$$a_j \rightarrow a_i^{\epsilon}{}_j^{j}.$$

Now let R be transformed into R^0 by a single move of type IB or type IC, and first let this be of type IB. Rearranging the notation, if necessary, let $\Sigma_i^0 = \Sigma_i$ if $i \neq j$. Then S_j^0, the image of Σ_j^0 in R, is the 2-sphere C^*, and we may assume that S_j^0 separates C_j, rather than C_j', from the rest of the 2-spheres C_i^ϵ, and that Σ_j^0 is oriented so that the positive side of S_j^0 is the one remote from C_j. Then the transformation $\Sigma_j \rightarrow \Sigma_j^0$, with due regard to orientation, can be regarded as the result of an isotopic deformation of Σ_j into Σ_j^0. Such a deformation is the resultant of deformations of the form

$$\Sigma \rightarrow \Sigma + \hat{E} \quad \text{(mod. 2)},$$

where $\hat{E}$ is the boundary of a 3-element E, which meets Σ in a 2-element f, on $\hat{E}$, and nowhere else. After a suitable subdivision and slight deformation of the circuits w, the intersections $\Sigma.w$ will be general after each stage of the deformation $\Sigma_j \rightarrow \Sigma_j^0$. Thus if (W) consists of cyclic words, or if (W) consists of true words and none of the 3-elements E contains O, the deformation $\Sigma_j \rightarrow \Sigma_j^0$ will determine an elementary transformation of (W), since the intersections $\Sigma \cdot w$ appear and disappear in consecutive pairs, those in each pair differing in sign. If (W) consists of true words and if some E contains O we modify the

[21] The transformation from U to U^* consists of replacing the vertices of C_j^ϵ by the corresponding vertices of $C_j^{-\epsilon}$.

deformation as follows. Let E^0 be a 3-element in E whose boundary meets $\hat{E}$ in a 2-element contained in f and nowhere else, and such that $\hat{E}^0$ meets each segment of w issuing from O in a single vertex, and otherwise does not meet w. For example, after a suitable subdivision of E, we may take

$$E^0 = N(s, E),$$

where s is a segment joining O to f and $N(L, K)$ stands for the aggregate of components in a complex K which meet L. Then we take the deformation

$$\Sigma \to \Sigma + \hat{E}$$

in two stages, namely

$$\Sigma \to \Sigma + \hat{E}^0$$

and

$$\Sigma + \hat{E}^0 \to \Sigma + \hat{E}^0 + (\hat{E} + \hat{E}^0).$$

The first stage determines an elementary transformation of the form

$$W_\lambda \to a_j^\epsilon W_\lambda a_j^{-\epsilon}$$

and the second stage also determines an elementary transformation. Thus in all cases

$$\Sigma \to \Sigma + \hat{E},$$

and hence the complete deformation

$$\Sigma_j \to \Sigma_j^0,$$

determines an elementary transformation of the words (W).

Now let R be transformed into R^0 by a single move of type IC. As before let S_j^0 cover C_j positively and let the detached region be bounded by S_j^0, C_i and C_j. Let μ stand for the move $R \to R^0$ and let μ^* be a move of type IC in which R is cut along a 2-sphere C^*, separating C_i and C_j from S_j^0. Then μ is the resultant of μ^*, followed by a move of type IB. After a suitable subdivision of R let

$$C^* = N(C_i + s + C_j, R),$$

where s is a segment joining C_i to C_j in the region bounded by C_i, C_j and S_j^0, and the positive side of C^* is the one remote from C_j. According to an argument in W. G. (p. 54) the move μ^* determines the simple transformation $a_i \to a_i a_j$ of the words (W), C^* being eventually renamed C_j. Therefore μ determines the resultant of the simple transformation $a_i \to a_i a_j$ followed by an elementary transformation.

In the general case the lemma now follows from induction on the number of moves in the transformation $R \to R^0$.

Let n_i be the number of times a_i and a_i' occur in (W). Then the circuits w pierce Σ_i in n_i vertices, and we shall prove the lemma:

LEMMA 3. *If (W) is a minimal set (T) and if C^* is a 2-sphere in R which separates C_k from C_k' then*

$$n^* \geqq n_k,$$

where the circuits w pierce Σ^*, the image of C^*, in n^* vertices.

Let C^* separate C_k^ϵ, and also $C_{i_1}^{\epsilon_1}, \cdots, C_{i_r}^{\epsilon_r}$ say, from $C_k^{-\epsilon}$, with the convention that $r = 0$ if C^* is isotopic to C_k^ϵ. If $r = 0$ the transformation of (W) determined by $\Sigma \to \Sigma^*$ is an elementary transformation. Otherwise let s_ρ be a segment in the region bounded by $C_k^\epsilon, C_{i_1}^{\epsilon_1}, \cdots, C_{i_r}^{\epsilon_r}$ and C^*, joining C_k^ϵ to $C_{i_\rho}^{\epsilon_\rho}$, and, after a suitable sub-division, let

$$C^{**} = N(C_k^\epsilon + C_{i_1}^{\epsilon_1} + \cdots + C_{i_r}^{\epsilon_r} + s_1 + \cdots + s_r, R),$$

oriented so that the positive side of C^{**} is the one remote from C_k. Let Σ^{**} be the image of C^{**} in M. As in the case of a single move of type IC (i.e. the case $r = 1$), it follows, from an obvious extension of the argument in W. G., that the transformation of (W) determined by $\Sigma \to \Sigma^{**}$ is the transformation (3.1). Therefore the transformation determined by $\Sigma \to \Sigma^*$ is (3.1) followed by an elementary transformation.

In either case, let

$$\Sigma_i^0 = \Sigma_i \text{ if } i \neq k, \qquad \Sigma_k^0 = \Sigma^*,$$

and let (W^0) be the set of words determined by the circuits w and the 2-spheres Σ^0. Then

$$L(W^0) = L(W) + n^* - n_k,$$

and we have only to show that $L(W) \leqq L(W^0)$.

If $r = 0$, $L(W) \leqq L(W^0)$ since $(W) \to (W^0)$ is an elementary transformation and (W) is reduced (E). If $r > 0$ let (W^*) be the set of words derived from (W) by the T-transformation consisting of (3.1) followed by a complete elementary reduction. Then $L(W^*) \leqq L(W^0)$, since (W^0) is derived from (W) by (3.1) followed by an elementary transformation which may or may not be a complete elementary reduction. Since (W) is minimal $L(W) \leqq L(W^*)$. Therefore in either case

$$L(W) \leqq L(W^0)$$

and the lemma is established.

Let (W) and (W^0) be two sets of words represented by the intersections of a set of circuits w^0 with sets of independent 2-spheres Σ and Σ^0. In W. G. (p. 53) I described a method for replacing Σ^0 by a set of surfaces having the same intersections with w^0 and having normal intersections with the 2-spheres Σ. That is to say each of the new surfaces S^0 meets a given one of the 2-spheres C_i^ϵ in a single circuit, if at all. Here we shall need the result stated in the lemma:

LEMMA 4. *The circuits w^0 can be replaced by a set of circuits whose intersections*

with Σ also represent the words (W) and, at the same time, Σ^0 can be replaced by a set of independent 2-spheres having normal intersections with Σ, whose intersections with the new circuits represent the words (W^0).

Unless the intersections $\Sigma \cdot \Sigma^0$ are already normal there is a surface S_β^0 such that two circuits of $\hat{S}_\beta^0$ are connected by a segment pq, on some C_i^ϵ, which does not meet any other circuit belonging to an[22] $\hat{S}^0$. Let E_1 and $E_2 (E_2 \supset q)$ be the 2-elements on C_i^ϵ which are bounded by these circuits of $\hat{S}_\beta^0$ and which do not contain pq. After a suitable sub-division let E_2^* be the 2-element on the boundary of $N(E_2, R)$ which is inside R and is bounded by the circuit on the boundary of $N(\hat{E}_2, S_\beta^0)$ which is inside R, and let pq^* be a segment on S_β^0 joining p to a vertex q^* on E_2^*. Let T be the solid tube in R bounded by

$$E_2 + E_2^* + N(\hat{E}_2, S_\beta^0).$$

We shall replace T by a 3-element T^*, defined as follows. Let R be sub-divided again in such a way that

$$F = N(E_2, T) \text{ and } F^* = N(E_2^*, T)$$

are 3-elements which do not meet each other. Let t consist of all the solids in R which meet $qp + pq^*$ and lie on the same side[23] of S_β^0 as pq. Then we assume the second sub-division of R to be such that

$$T^* = F + t + F^*$$

is a 3-element. We map T semi-linearly on T^* in such a way that each vertex of E_2 and E_2^* corresponds to itself. If one of the segments in R corresponding to a segment of some w pierces E_2 it will also pierce E_2^* without leaving T, because of our understanding that all intersections are general. Similarly any portion of an S^0 which lies inside T is a cylinder bounded by circuits in E_2 and E_2^*. We replace these segments and cylinders[24] by their images in T^*, and S_β^0 by

$$S_\beta^0 + \hat{T} + \hat{T}^*,$$

which is a surface meeting C_i^ϵ in the same circuits as S_β^0 and also in a strip bordering pq, since addition is mod. 2. The corresponding alteration in M substitutes other 2-spheres for any of the 2-spheres Σ^0 which have been modified. For we have not altered the topology of these surfaces by substituting cylinders internal to T^* for cylinders internal to T. In altering T itself we have merely removed the cylinder bounded by $\hat{E}_2$ and $\hat{E}_2^*$, deformed $\hat{E}_2^*$ by removing a strip on S_β^0 bordering pq^* and then restored a cylinder bounded by $\hat{E}_2$ and the image of $\hat{E}_2^*$ in the deformation. Since all the chains have general intersections neither the

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[22] W. G., p. 53.

[23] The surfaces S^0 are two-sided, since they correspond to regions on the 2-spheres Σ^0, and E_2 and E_2^* lie on the same side of S_β^0, pq and t lying on the other side.

[24] Though addition is mod. 2 we shall sometimes use the minus sign to make the formulae more explicit.

first cylinder nor the strip bordering pq^* meets any of the segments corresponding to the circuits w. Any segments which previously ran through T are now replaced by segments in T^* with the same end points. Thus the circuits w and the 2-spheres Σ^0 are replaced by new sets of circuits and 2-spheres, say w^* and Σ^*, such that the intersections $\Sigma \cdot w^*$ and $\Sigma^* \cdot w^*$ are the same as $\Sigma \cdot w$ and $\Sigma^0 \cdot w$ respectively.

Any one of the cylinders on a surface S^0 which ran through T, together with 2-elements on E_2 and E_2^*, bounded a 3-element T_λ, in T, and the cylinder in T^* which replaces it, together with the same 2-elements, bounds a 3-element T_λ^*, in T^*. Let $T_0 = T$, $T_0^* = T^*$ and let τ_λ and τ_λ^* be the respective images of T_λ and T_λ^* in $M(\lambda = 0, \cdots, n)$. Then either

$$\Sigma_k^* = \Sigma_k^0$$

or

$$\Sigma_k^* = \Sigma_k^0 + \hat{\tau}_0 + \hat{\tau}_0^* + \cdots + \hat{\tau}_n + \hat{\tau}_n^*.$$

Therefore

$$\Sigma_k^* \sim \Sigma_k \qquad (k = 1, \cdots, p),$$

and the 2-spheres Σ^* are independent.

The circuits $\hat{E}_1$ and $\hat{E}_2$ may now be united by a slight deformation which carries the strip bordering pq through C_i^ϵ into the neighbourhood of $C_i^{-\epsilon}$, and the lemma follows from induction on the total number of circuits in the boundaries[25] $\hat{S}_1^0, \cdots, \hat{S}_q^0$.

As a final lemma we prove the converse of lemma 2, restricting ourselves to the case where (W) and (W^0) are reduced (E).

LEMMA 5. *Any two equivalent sets of words* (W^0) *and* (W), *both reduced* (E), *can be simultaneously represented by the intersections of a single set of circuits* w, *with sets of independent 2-spheres* Σ^0 *and* Σ.

Let (W^0) and (W) be equivalent sets of words, both reduced (E), and let (W^0) be represented in M by the intersections of a set of circuits w^*, with a set of independent 2-spheres Σ^0. From an argument in W. G., §1, it follows that the transformation $(W^0) \rightarrow (W)$ can be exhibited in the form

$$(W) = Ut_\mu \cdots t_1(W^0),$$

where t_ρ is a simple transformation of the form $x \rightarrow xy$, not followed by any elementary reductions, and U consists only of elementary reductions, (W) being reduced (E), and transformations of the type $x \rightarrow x'$. As in W. G. (p. 54), it follows from induction on μ that

$$(W^*) = t_\mu \cdots t_1(W^0)$$

[25] From the existence of the original segment pq, it follows that the relation 'having normal intersection with' is symmetric between Σ and Σ^0.

is represented in M by the intersections of the original circuits w^* with a set of independent 2-spheres Σ. In the presence of lemma 4 we lose no generality in assuming the intersections $\Sigma \cdot \Sigma^0$ to be normal, which we accordingly do.

As in W. G. (p. 54), where (W^0) is an independent set of letters, we have to show that the reduction $(W^*) \to (W)$ can be copied by a deformation of the circuits w^* into a set whose intersections with Σ^0 and Σ represent (W^0) and (W) respectively. Let PQ be the segment in R corresponding to a syllable xx' in (W^*), say $a_i a_i'$. Since the intersections $\Sigma^0 \cdot \Sigma$ are normal, P and Q are joined by a segment s, on C_i, which does not meet any one $\hat{S}_\beta^0$ more than once, and PQ can be deformed[26] into s in R. As we copy this deformation in M the intersections $w^* \cdot \Sigma^0$ may be modified. If so it follows from an argument similar to one used in proving lemma 2 that the new intersections $w^* \cdot \Sigma^0$ will represent a set of words which are derived from (W^0) by elementary reductions and insertions. But an argument used in W. G. shows that the number of intersections $s \cdot \hat{S}^0$ does not exceed the number of intersections $PQ \cdot S^0$. That is to say, the number of intersections $w^* \cdot \Sigma^0$, and therefore the number of letters in (W^0), is not increased. Since (W^0) was originally reduced (E) it follows that the set of circuits w^* continues to represent (W^0) after the deformation. As in W. G., the elementary reduction which consists of omitting $a_i a_i'$ may now be copied in R by deforming s through C_i into the neighbourhood of C_i'. Repeating this process, w^* may be replaced by a set of circuits w, whose intersections with Σ^0 and Σ represent (W^0) and (W) respectively. Thus the lemma is established.

We are now ready to prove:

THEOREM 3. *Any two equivalent minimal sets (T) are interchangeable by level T-transformations.*

The inverse of a T-transformation, applied to a set which is reduced (E), is a T-transformation. Therefore the inverse of a level transformation is a level transformation and it is enough to show that two equivalent minimal sets (W^0) and (W) can be identified by level transformations applied indiscriminately to either or both of them.

Let (W^0) and (W) be represented in M by the intersections of w with independent sets of 2-spheres Σ^0 and Σ. First let us assume that each Σ^0 either coincides with some Σ or does not meet any Σ. Let the notation be such that

$$\Sigma_1^0, \cdots, \Sigma_k^0$$

do not meet any Σ, while

$$\Sigma_\rho^0 = \Sigma_\rho \text{ if } \rho > k$$

(possibly $k = p$, in which case no Σ^0 meets a Σ). If k is zero $(W) = (W^0)$, and we shall use induction on k.

We have

$$\Sigma_\rho^0 \sim r_{\rho j} \Sigma_j \qquad \text{(summed for } j = 1, \cdots, p),$$

[26] We need not concern ourselves with the fact that PQ may cross itself or other segments of $g(w^*)$ during this deformation.

where $r_{\rho j}$ are calculated mod. 2. Also $|\, r_{\rho\sigma} \,| = 1$ $(\rho, \sigma = 1, \cdots, k)$, since the Σ^0's are independent. Rearranging the notation, if necessary, we may suppose $r_{11} = \cdots = r_{kk} = 1$. Then S^0_ρ, the image of Σ^0_ρ in R, separates C_ρ from C'_ρ, and since (W) is a minimal set it follows from lemma 3 that

$$n^0_\rho \geqq n_\rho,$$

where n^0_i is the number of intersections $\Sigma^0_i \cdot w$. Since $n^0_i = n_i$ if $i > k$ we have

$$L(W^0) = n^0_1 + \cdots + n^0_p \geqq n_1 + \cdots + n_p = L(W).$$

Similarly

$$L(W) \geqq L(W^0)$$

and it follows that

$$L(W) = L(W^0).$$

Therefore[27] $n^0_\rho = n_\rho$. We have seen that the transformation of the words (W) determined by the substitution $\Sigma_k \to \Sigma^0_k$ can be exhibited either as an elementary transformation U, or as a transformation of the form (3.1) followed by an elementary transformation U. In the first case U is the identity, since $n^0_k = n_k$ and the words (W) are reduced (E). In the second case, U is a complete elementary reduction since the set (W) is minimal. Otherwise the T-transformation determined by (3.1) would be a T-reduction, since $n^0_k = n_k$ and any elementary reduction applied after U would reduce the number of letters in the new words (W). Therefore $\Sigma_k \to \Sigma^0_k$ is a T-transformation. Therefore it is a level transformation and this special case of the theorem follows from induction on k.

Returning to our general rule that all intersections are to be general relative to slight deformations, we shall complete the proof by induction on the number of circuits in which Σ and Σ^0 intersect. The process of normalization used in proving lemma 4 reduces the number of circuits in the intersections $\Sigma \cdot \Sigma^0$. Therefore we may assume the latter to be normal.

Let S^0_β be any bounded S^0 and let

$$\hat{S}^0_\beta = c + c'_1 + \cdots + c'_r$$

where c is on C^ϵ_i and c'_ρ is on $C^{-\epsilon_\rho}_{i_\rho}$ (possibly $C^{-\epsilon_\rho}_{i_\rho} = C^{-\epsilon}_i$ for some ρ). Let E be one of the 2-elements into which C^ϵ_i is separated by c, let E'_ρ be one of the 2-elements into which $C^{-\epsilon_\rho}_{i_\rho}$ is separated by c'_ρ and let

$$F = S^0_\beta + E'_1 + \cdots + E'_r.$$

Let $M(K)$ stand for the image in M of any complex K in R, let Σ^0_j be the Σ^0 containing $M(S^0_\beta)$ and let E^0 be the 2-element on Σ^0_j which is bounded by $M(c)$ and contains $M(S^0_\beta)$. Using induction on the number of circuits belonging to

[27] Cf. J. N., p. 91.

$\Sigma \cdot \Sigma_j^0$ which are inside E^0, we shall define the *proper* one of the two 2-elements into which C_i^ϵ is separated by c. If

$$M(C_i^\epsilon - E + F) \sim x_t \Sigma_t, \qquad M(E + F) \sim y_t \Sigma_t \tag{3.2}$$

(summed over t), we have $x_i + y_i = 1$, the i being the same as in C_i^ϵ. If E^0 does not meet any Σ except in its boundary, $\hat{S}_\beta^0$ consists of a single circuit. Otherwise we choose E'_ρ on $C_{i_\rho}^{-\epsilon_\rho}$ such that its image E_ρ, on $C_{i_\rho}^{\epsilon_\rho}$, is the proper 2-element bounded by c_ρ, the image of c'_ρ. In either case the proper 2-element E_i on C_i^ϵ, is to be the one such that the y_i of (3.2) is zero and $x_i = 1$.

If k is the number of intersections $M(E) \cdot w$, E being the proper 2-element of the last paragraph, and if k^0 is the number of intersections $E^0 \cdot w$, I say that

$$k \leqq k^0.$$

For let k_ρ be the number of intersections $M(E_\rho) \cdot w$ and k_ρ^0 the number of intersections $E_\rho^0 \cdot w$, where E_ρ^0 is the 2-element in E^0 bounded by $M(c_\rho)$. Then

$$k^0 = s^0 + k_1^0 + \cdots + k_r^0, \tag{3.3}$$

where s^0 is the number of intersections $M(S_\beta^0) \cdot w$. Since

$$M(C_i^\epsilon - E + F) \sim \Sigma_i + \cdots$$

it follows that the 2-sphere $C_i^\epsilon - E + F$, after a slight deformation in R, separates C_i from C'_i. The number of intersections $M(C_i^\epsilon - E + F) \cdot w$ is $n_i - k + s^0 + k_1 + \cdots + k_r$, and it follows from lemma 3 that

$$k \leqq s^0 + k_1 + \cdots + k_r. \tag{3.4}$$

If $\hat{S}_\beta^0$ consists of the single circuit c, $k_\rho = 0$ and $k^0 = s^0$. Otherwise, assuming $k_\rho \leqq k_\rho^0$, the required result follows from (3.4), (3.3) and induction on the number of circuits of $\Sigma \cdot \Sigma_j^0$ inside E^0.

Let Σ_i be one of the 2-spheres Σ which meets a 2-sphere in the set Σ^0. Let c be an 'extreme' circuit belonging to $\Sigma_i \cdot \Sigma^0$, meaning that one of the 2-elements into which Σ_i is separated by c contains no other circuit belonging to $\Sigma_i \cdot \Sigma^0$. Let A be the 2-element in question and let E^0 be the 2-element bounded by c on Σ_j^0, the Σ^0 which contains c, such that $x_j = 1$ in the homology

$$\Sigma_j^0 - E^0 + A \sim x_t \Sigma_t^0.$$

Now cut M along the 2-spheres Σ^0 to form the model R^0, referred to in lemma 1. The image in R^0 of $\Sigma_j^0 - E^0 + A$, slightly deformed, is a 2-sphere which separates the two images of Σ_j^0 from each other. The number of intersections $(\Sigma_j^0 - E^0 + A) \cdot w$ is $n_j^0 - k^0 + l$, where k^0 is the number of intersections $E^0 \cdot w$ and l is the number of intersections $A \cdot w$. Therefore

$$l \geqq k^0,$$

according to lemma 3. If $l = k^0$ it follows from a previous argument that the substitution $\Sigma_j^0 \rightarrow \Sigma_j^0 - E^0 + A$ determines a level transformation of (W^0).

Since it reduces the number of intersections $\Sigma \cdot \Sigma^0$ the theorem, in this case, follows from the hypothesis of the induction. If $l > k^0$ let C_i^ϵ be the image in R of the side of Σ_i incident with E^0. If $R(A)$, the image of A in R, were the proper 2-element bounded on C_i^ϵ by $R(c)$ we should have $l \leqq k^0$. Therefore

$$E = C_i^\epsilon - R(A)$$

is the proper 2-element, and if k is the number of intersections $M(E) \cdot w$ we have $k \leqq k^0 < l$. That is to say, more than half the intersections $\Sigma_i \cdot w$ lie in A unless the substitution $\Sigma_j^0 \to \Sigma_j^0 - E^0 + A$ determines a level transformation of (W^0).

But the 2-element $\Sigma_i - A$ also contains a 2-element A_1, which is bounded by a circuit of $\Sigma_i \cdot \Sigma^0$ and contains no other circuit of $\Sigma_i \cdot \Sigma^0$. Adding suitable subscripts to indicate that A is replaced by A_1 we see that one or other of the substitutions

$$\Sigma_j^0 \to \Sigma_j^0 - E^0 + A, \qquad \Sigma_{j_1}^0 \to \Sigma_{j_1}^0 - E_1^0 + A_1$$

determines a level transformation of (W^0). They both reduce the number of circuits in $\Sigma \cdot \Sigma^0$, and the theorem is established.

From this theorem it is clear that the process described earlier in this section will exhibit the equivalence of equivalent sets. As before, let n_i be the number of times a_i and a_i' appear in (W) and let q be the number of n's which are not zero. Then $L(W)$, q and n_i are each unaltered by level T-transformations and it follows that these numbers, calculated for a minimal (T) set (W), are invariants of the family of equivalent sets. They are analogous to $L(\alpha)$, m and $k_\lambda = l(\alpha_\lambda)$, calculated for a minimal (N) set (α).

4. In this section we indicate how the results of §§2 and 3 can be applied to the study of free factor groups of free groups. Let Γ be the free group generated by $\alpha_1, \cdots, \alpha_m$ and let γ be the factor group Γ/Γ_0, where Γ_0 is the smallest invariant sub-group containing

$$R_1(\alpha, \alpha'), \cdots, R_q(\alpha, \alpha').$$

Let $\alpha_\lambda \to \alpha_\lambda(a, a')$ define a simple isomorphism between γ and the free group G, generated by $a_1, \cdots, a_p$. Then

$$R_\lambda[\alpha(a, a'), \alpha'(a, a')] = 1, \tag{4.1}$$

an element $F(\alpha, \alpha')$, of Γ, belongs to Γ_0 if

$$F[\alpha(a, a'), \alpha'(a, a')] = 1,$$

and there are elements $F_i(\alpha, \alpha')$ such that

$$a_i = F_i[\alpha(a, a'), \alpha'(a, a')].$$

Therefore $\alpha_1(a, a'), \cdots, \alpha_m(a, a')$ constitute a basis for G, and there is a basis for Γ, say $\beta_1(\alpha, \alpha'), \cdots, \beta_m(\alpha, \alpha')$, such that the set consisting of the corresponding elements in G, namely

$$\beta_\lambda(a, a') = \beta_\lambda[\alpha(a, a'), \alpha'(a, a')],$$

is reduced (N). Re-arranging the notation, if necessary, it follows from theorem 1 that

$$\begin{cases} \beta_i(a, a') = a_i & (i = 1, \cdots, p), \\ \beta_\rho(a, a') = 1 & (\rho = p+1, \cdots, m), \end{cases} \tag{4.2}$$

m exceeding[28] p unless the relations $R_\sigma = 1$ are vacuous. Then γ is the subgroup freely generated by $\beta_1, \cdots, \beta_p$, and the relations

$$S_1(\beta, \beta') = \cdots = S_q(\beta, \beta') = 1 \tag{4.3}$$

are equivalent to

$$\beta_{p+1} = \cdots = \beta_m = 1, \tag{4.4}$$

where

$$S_\sigma(\beta, \beta') = R_\sigma[\alpha(\beta, \beta'), \alpha'(\beta, \beta')].$$

In particular, let γ be defined by a single relation

$$R(\alpha, \alpha') = 1,$$

where $R \not\equiv 1$ when the word R is reduced (E, α). With the notation used in the last paragraph we see by considering the free Abelian group associated with Γ that $p = m - 1$, since $p < m$ and the relations (4.4) are equivalent to (4.3), with $q = 1$. Moreover it follows from the equivalence of (4.3) and (4.4) that[29]

$$S(\beta, \beta') \equiv \beta_m^\epsilon,$$

when the cyclic word S is reduced (E, β). That is to say the cyclic word $R(\alpha, \alpha')$ is simple.[30]

Conversely, if R is a simple word it is obvious that γ is a free group, and we have the theorem:

THEOREM 4. *The group generated by $\alpha_1, \cdots, \alpha_m$, subject to a single relation*

$$R(\alpha, \alpha') = 1,$$

which is not vacuous, is a free group if, and only if, the cyclic word $R(\alpha, \alpha')$ is simple.

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[28] Cf. F. Levi, *Math. Zeit.*, 37 (1933), 90–97.

[29] W. Magnus, *Journal für die r.u.a. Math.* 163 (1930), 142.
See also K. Reidemeister, *Einführung in die kombinatorische Topologie*, Brunswick (1932), p. 92.

[30] W. G., §1.

It follows from this theorem that one can tell whether or no such a group is a free group by the process of reduction (T).

5. A problem which remains open is to determine whether or no Γ is equivalent to Γ^* under the automorphisms of G, where Γ and Γ^* are sub-groups of the latter, generated by given sets of elements (α) and (α^*). If Γ and Γ^*, generated by finite sets (α) and (α^*), are equivalent, one might hope to identify them by sequences of reductions (N) and (T), applied to (α) and (α^*), followed by level transformations. The following example shows that one cannot always do this

Let

$$\alpha_1 = (ab^{-1})^2\, b^{-2}(ab^{-1})^2\, a^3, \qquad \beta_1 = a^{-3}b^{-5},$$

and

$$\alpha_2 = a^2b^{-2}a^2b^{-5}, \qquad \beta_2 = (ab)^{-3}\, b^{-5}.$$

Then (α_1, β_1) and (α_2, β_2) are both unassailable by reductions of either kind. Since $L(\alpha_1, \beta_1) = 21$ and $L(\alpha_2, \beta_2) = 22$ they are not interchangeable by level transformations. All the same, if

$$\alpha = a^2b^{-2}a^2(ab)^3, \qquad \beta = (ab)^{-3}\, b^{-5},$$

$(\alpha_1, \beta_1) \rightarrow (\alpha, \beta)$ in the automorphism $(a \rightarrow ab, b \rightarrow b)$, while $(\alpha_2, \beta_2) = (\alpha\beta, \beta)$

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ON THE COMPLETENESS OF LAMBERT FUNCTIONS, II[1]

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1. **Introduction.** In an earlier note by the same title[2] the authors have investigated completeness properties of systems of functions of the form $S(\alpha, \beta, \lambda)$:

$$(1.1) \qquad (1-t)^{\alpha} k_{\beta}(t^{\lambda_n}),$$

where $n = 1, 2, 3, \cdots, 0 < t < 1$, and

$$(1.2) \qquad k_{\beta}(w) = \sum_{1}^{\infty} m^{\beta} w^{m}.$$

Their results could be summed up in the following

THEOREM 1. *Let* $\Re(\lambda_n) \geqq \delta > 0$, $\sum_1^{\infty} \Re(1/\lambda_n) = +\infty$. *The system* $S(\alpha, \beta, \lambda)$ *is complete in* $L_p(0, 1)$, p *fixed*, $1 \leqq p < \infty$, *if* $\alpha > \max(\beta + 1 - 1/p, -1/p)$, *and* $1 + S(\alpha, \beta, \lambda)$ *is complete in* $C[0, 1]$ *if* $\alpha > \max(\beta + 1, 0)$.

These results admit of various generalizations which will be given in the present note. A set of elements is said to be *complete* in a (Banach) space if the only linear functional defined all over the space which vanishes for all elements of the set is the zero functional. Using this definition we associated with the given set S a certain analytic function $F(z)$, holomorphic in the right half plane. If the set is incomplete in the space in question, it is possible to choose $F(z)$ so that it vanishes at $z = \lambda_n$ for all n; the generalized theorem of Blaschke being applicable, $F(z)$ vanishes identically. On the other hand, $F(z)$ turned out to be expressible in the form

$$F(z) = \sum_{1}^{\infty} m^{\beta} L(mz),$$

where $L(z)$ is a Laplace transform, and conversely $L(z)$ was expressible linearly in terms of $F(nz)$ with the aid of the special inversion formula of Möbius, the inversion being unique within the class of functions appropriate to the problem. Hence $F(z) \equiv 0$ implies $L(z) \equiv 0$, and from this we could conclude the completeness by easy steps.

The same general method applies without essential change to much more general systems. Thus in §2 below we replace $(1-t)^{\alpha}$ by an arbitrary weight function $\varphi(t)$. In §3 we replace in addition the coefficients m^{β} in the generating function $k_{\beta}(w)$ by arbitrary complex numbers a_m such that $a_1 = 1$, $|a_m| \leqq Bm^{\beta}$. This change is of more fundamental nature than one would think a priori, be-

[1] Presented to the American Mathematical Society April 10, 1936.
[2] Bulletin Amer. Math. Soc., 42 (1936), 411–418.

cause we have now to resort to the general form of the inversion formula of Möbius, and the restrictions on the parameters have to be made more severe than in the special case. In §4 we take one further step by replacing the generating power series by an irregular power series whose exponents form a multiplicative system. On the other hand, the mere replacement of the exponents $\{m\}$ and the coefficients $\{m^\beta\}$ by other multiplicative systems does not call for any other tools than the special inversion formula. In §5 we discuss the completeness in $C[0, 1]$ of the system $1 + S(\beta + 1, \beta, \lambda)$ which requires a refinement of the method.

2. **General weight functions.** Let $\varphi(t)$ be a measurable function such that $t^\eta\varphi(t) \in L_1(0, 1)$ for every $\eta > 0$, and $\varphi(t) = 0$ at most on a set of measure zero. We set

$$M(x) = \int_0^1 t^x |\varphi(t)|\, dt, \tag{2.1}$$

and

$$\liminf_{x\to\infty} \frac{\log [M(x)]^{-1}}{\log x} = \gamma. \tag{2.2}$$

Here $0 \leqq \gamma \leqq \infty$. We assume $\gamma < \infty$ in the following. Then to any $\epsilon > 0$ there exists a $\xi = \xi(\epsilon)$ such that

$$M(x) < x^{-\gamma+\epsilon} \text{ for } x > \xi, \tag{2.31}$$

$$M(x) > x^{-\gamma-\epsilon} \text{ for infinitely many } x = x_\nu \to +\infty. \tag{2.32}$$

We can characterize γ by other properties as well, which is shown by the following lemmas.

LEMMA 2.1. *The exponent of convergence of the series*

$$\sum_1^\infty [M(nx_0)]^\sigma, \qquad x_0 > 0,$$

equals $1/\gamma$, *independently of* x_0.

PROOF. The convergence of the series for $\sigma > 1/\gamma$ follows from (2.31). Conversely, if the series is convergent,

$$n^{1/\sigma} M(nx_0) \to 0,$$

since the terms are steadily decreasing. This implies that $M(x) = o(x^{-1/\sigma})$ as $x \to \infty$, since $M(x)$ is monotone. Hence $\sigma \geqq 1/\gamma$ by (2.32).

LEMMA 2.2. *Putting*

$$\psi(t) = \int_t^1 |\varphi(u)|\, du,$$

we have

$$\gamma = \liminf_{t\to 1} \frac{\log [\psi(t)]^{-1}}{\log (1-t)^{-1}}.$$

PROOF. We have

$$\psi(t) < \int_t^1 \left(\frac{u}{t}\right)^{\frac{1}{1-t}} |\varphi(u)|\, du = O\left\{M\left(\frac{1}{1-t}\right)\right\} = O[(1-t)^{\gamma-\epsilon}].$$

Conversely, integrating by parts,

$$\begin{aligned}\int_t^1 u^x |\varphi(u)|\, du &= -[u^x\psi(u)]_t^1 + x\int_t^1 u^{x-1}\psi(u)\, du \\ &= t^x\psi(t) + x\int_t^1 u^{x-1}\psi(u)\, du \\ &\to M(x) \text{ as } t \to 0.\end{aligned}$$

Hence, for $x > 0$, $t^x\psi(t) \to 0$ as $t \to 0$, and

$$M(x) = x\int_0^1 u^{x-1}\psi(u)\, du = O\left\{x\int_0^1 u^{x-1}(1-u)^{\gamma-\epsilon}\, du\right\} = O(x^{-\gamma+\epsilon}).$$

These estimates prove the lemma.

Let us write $S(\varphi(t), \beta, \lambda)$ for the system of functions

$$\varphi(t)k_\beta(t^{\lambda_n}), \qquad n = 1, 2, 3, \cdots$$

where $k_\beta(w)$ is defined by (1.2). We shall prove a generalization of that part of Theorem 1 which refers to completeness in $L_1(0, 1)$. Our assumptions on $\varphi(t)$ are adapted to this case, but the reader who is interested in completeness in $L_p(0, 1)$ or in $C[0, 1]$ will have no difficulties in modifying these assumptions so as to suit his needs.

THEOREM 2. *Let $\varphi(t)$ satisfy the assumptions stated above. Let $\Re(\lambda_n) \geqq \delta > 0$, $\sum_1^\infty \Re(1/\lambda_n) = +\infty$. Then $S(\varphi(t), \beta, \lambda)$ is complete in $L_1(0, 1)$ provided $\gamma > \beta + 1$.*

PROOF. We follow the procedure employed in the proof of Theorem 1. It is necessary as well as sufficient to show that the only bounded measurable function which is orthogonal to all functions of the system S, is equivalent to zero. We form

$$F(z) = \int_0^1 k_\beta(t^z)\varphi(t)g(t)\, dt, \qquad \Re(z) > 0, \tag{2.4}$$

with an arbitrary bounded measurable function $g(t)$. If $|g(t)| \leqq g$, then

$$|F(z)| \leqq g\sum_1^\infty m^\beta M(mx) \leqq gM_\epsilon x^{-\gamma+\epsilon}\sum_1^\infty m^{\beta-\gamma+\epsilon}, \tag{2.5}$$

for $\epsilon > 0$, $x > \delta(\epsilon)$, and a suitably chosen constant M_ϵ. The series converges for a sufficiently small ϵ since $\gamma > \beta + 1$. It follows that the functions of S belong to $L_1(0, 1)$. Further, $F(z)$ is holomorphic in the right half plane and bounded for $x \geqq \delta > 0$. If the system is not complete, there exists a $g(t)$ such that the corresponding function $F(z)$ vanishes for $z = \lambda_n$, $n = 1, 2, 3, \cdots$. By

cause we have now to resort to the general form of the inversion formula of Möbius, and the restrictions on the parameters have to be made more severe than in the special case. In §4 we take one further step by replacing the generating power series by an irregular power series whose exponents form a multiplicative system. On the other hand, the mere replacement of the exponents $\{m\}$ and the coefficients $\{m^\beta\}$ by other multiplicative systems does not call for any other tools than the special inversion formula. In §5 we discuss the completeness in $C[0, 1]$ of the system $1 + S(\beta + 1, \beta, \lambda)$ which requires a refinement of the method.

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$$\to M(x) \text{ as } t \to 0.$$

Hence, for $x > 0$, $t^x\psi(t) \to 0$ as $t \to 0$, and

$$M(x) = x \int_0^1 u^{x-1}\psi(u)\, du = O\left\{x \int_0^1 u^{x-1}(1-u)^{\gamma-\epsilon}\, du\right\} = O(x^{-\gamma+\epsilon}).$$

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for $\epsilon > 0$, $x > \delta(\epsilon)$, and a suitably chosen constant M_ϵ. The series converges for a sufficiently small ϵ since $\gamma > \beta + 1$. It follows that the functions of S belong to $L_1(0, 1)$. Further, $F(z)$ is holomorphic in the right half plane and bounded for $x \geqq \delta > 0$. If the system is not complete, there exists a $g(t)$ such that the corresponding function $F(z)$ vanishes for $z = \lambda_n$, $n = 1, 2, 3, \cdots$. By

the extension of Blaschke's theorem to the half plane, $F(z)$ is identically zero. On the other hand

$$F(z) = \sum_1^\infty m^\beta L(mz), \tag{2.6}$$

where

$$L(z) = \int_0^1 t^z \varphi(t) g(t)\, dt, \tag{2.7}$$

$$| L(z) | \leqq g M_\epsilon x^{-\gamma+\epsilon}, \qquad x \geqq \delta(\epsilon) > 0. \tag{2.8}$$

But if $L(z)$ satisfies (2.8) then it can be expressed linearly in terms of the set of functions $F(nz)$ with the aid of the inversion formula of Möbius. Let $\mu(n)$ be the factors of Möbius. Then the series

$$\sum_1^\infty \mu(n) n^\beta F(nz) = \sum_1^\infty \sum_1^\infty \mu(n)(mn)^\beta L(mnz) \tag{2.9}$$

is absolutely convergent, being dominated by the series

$$g M_\epsilon x^{-\gamma+\epsilon} \sum_1^\infty \sum_1^\infty (mn)^{\beta-\gamma+\epsilon},$$

which is convergent by assumption. We can then rearrange the double series in (2.9) and reduce with the aid of the properties of $\mu(n)$. The result is that

$$L(z) = \sum_1^\infty \mu(n) n^\beta F(nz) \tag{2.10}$$

is the only function satisfying (2.6) and (2.8). In our case $F(z) \equiv 0$, so that $L(z) \equiv 0$, and by (2.7) this implies $\varphi(t)g(t) \sim 0$ or $g(t) \sim 0$ since $\varphi(t)$ vanishes only on a null set. Hence we have proved the completeness of S in L_1.

The condition $\gamma > \beta + 1$ cannot be improved upon, because if $\gamma = \beta + 1$, the set S may cease to belong to L_1, so that the completeness problem becomes meaningless. This is shown by taking $\varphi(t) = (1 - t)^\alpha$, $\alpha > -1$, i.e., the special case of Theorem 1. Here $\gamma = \alpha + 1$, and if $\alpha = \beta$, none of the functions $(1 - t)^\alpha k_\beta(t^{\lambda_n})$ is in $L_1(0, 1)$.

3. **Other generating power series.** The work of our first note as well as that of the preceding paragraph is based upon the special inversion formula of Möbius. This is due to the structure of the generating function $k_\beta(w)$, namely to the fact that the coefficient of w^m is precisely m^β. If the coefficient is merely $O(m^\beta)$ instead, the situation changes, and the general form of the inversion formula must be used. It will be sufficient to consider the case $p = 1$.

THEOREM 3. *Let $\varphi(t)$ and $\{\lambda_n\}$ satisfy the assumptions of §2. Let $a_1 = 1$, $| a_m | \leqq B m^\beta$, $m = 2, 3, \cdots$. Let $S(\varphi(t), a, \lambda)$ denote the system of functions*

$$\varphi(t) \sum_{m=1}^\infty a_m t^{m\lambda_n}, \qquad n = 1, 2, 3, \cdots \tag{3.1}$$

$S(\varphi(t), a, \lambda)$ is complete in $L_1(0, 1)$ if $\gamma > \beta + \rho(B)$, where $\rho(B)$ is the positive root of the equation

$$\zeta(s) = 1 + \frac{1}{B}. \tag{3.2}$$

For $\gamma = \beta + \rho(B)$, there exists a choice of $\varphi(t)$ and $\{a_m\}$ satisfying the above conditions, such that the corresponding system $S(\varphi(t), a, \lambda)$ is in $L_1(0, 1)$ without being complete in the space.

REMARK. We note that the equation (3.2) has one and only one root >1 for every fixed positive B, and that $\rho(B)$ is a monotone increasing function of B which tends to 1 or to ∞ according as B tends to 0 or ∞.

PROOF. We proceed as above. The function $F(z)$ is defined by

$$F(z) = \sum_1^\infty a_m \int_0^1 t^{mz}\varphi(t)g(t)\,dt, \qquad x > 0. \tag{3.3}$$

Formula (2.5) is still valid. Since $\rho(B) > 1$ the dominant series converges, so that $S(\varphi(t), a, \lambda) \subset L_1(0, 1)$, and $F(z)$ is holomorphic in $x > 0$ as well as bounded for $x \geqq \delta > 0$. We conclude as above that if S is not complete in L_1, we can find a $g(t)$, not equivalent to zero, such that $F(\lambda_n) = 0$ for every n, and consequently $F(z) \equiv 0$. On the other hand

$$F(z) = \sum_1^\infty a_m L(mz), \tag{3.4}$$

where $L(z)$ is defined by (2.7) and satisfies (2.8).

At this juncture a new element enters into the proof. Let b_n, $n = 1, 2, 3, \cdots$ be determined by the algorithm of Möbius[3]

$$\begin{aligned} a_1 b_1 &= 1, \\ \sum_{d|n} a_d b_{n/d} &= 0. \qquad n = 2, 3, \cdots \end{aligned} \tag{3.5}$$

The b's are uniquely determined and are given by the following expression

$$b_n = \sum (-1)^{\alpha_1+\alpha_2+\cdots} C_{\alpha_1\alpha_2\cdots} a_{d_1}^{\alpha_1} a_{d_2}^{\alpha_2} \cdots. \tag{3.6}$$

Here the d's run through all the divisors of n which are >1 and $n = d_1^{\alpha_1} d_2^{\alpha_2} \cdots$; $C_{\alpha_1\alpha_2\cdots}$ is the combinatorial function which gives the number of arrangements of an aggregate consisting of α_1 objects of one kind, α_2 objects of another, $\cdots$, order being essential between elements belonging to different classes but not within the class.

Let us now form the series

$$\sum_1^\infty b_n F(nz) = \sum_{n=1}^\infty b_n \sum_{m=1}^\infty a_m L(mnz). \tag{3.7}$$

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[3] See A. F. Möbius, Journal für die reine und angewandte Mathematik, 9 (1832) 105–123, and *Gesammelte Werke*, IV, pp. 591–612. Parts of this paper are reproduced in Netto's *Lehrbuch der Combinatorik*, 2nd ed. pp. 172–180. Formula (3.6) below is easily proved by complete induction.

The double series is absolutely convergent if

$$\sum_1^\infty \sum_1^\infty (mn)^{-\gamma+\epsilon} |a_m| |b_n| \tag{3.8}$$

converges. But (3.6) shows that

$$\begin{aligned} |b_n| &\leqq \sum C_{\alpha_1\alpha_2\cdots} |a_{d_1}|^{\alpha_1} |a_{d_2}|^{\alpha_2} \cdots \\ &\leqq \sum C_{\alpha_1\alpha_2\cdots} B^{\alpha_1+\alpha_2+\cdots} d_1^{\beta\alpha_1} d_2^{\beta\alpha_2} \cdots \\ &= n^\beta \sum C_{\alpha_1\alpha_2\cdots} B^{\alpha_1+\alpha_2+\cdots}, \end{aligned} \tag{3.9}$$

where the summation extends over the various factorizations of n. Hence (3.8) is dominated by

$$\sum_{n=1}^\infty \sum_{m=1}^\infty (mn)^{\beta-\gamma+\epsilon} \sum C_{\alpha_1\alpha_2\cdots} B^{\alpha_1+\alpha_2+\cdots}. \tag{3.10}$$

Since $\alpha > \beta + 1$, the summation with respect to m can be carried out and gives the factor $\zeta(\gamma - \beta - \epsilon)$. In the remaining summation with respect to n we first isolate the term corresponding to $n = 1$, this term can be taken to be unity since $b_1 = 1$ and the estimate yielded by (3.9) is vacuous. Collecting terms in which $\alpha_1 + \alpha_2 + \cdots$ has a constant value, we can rewrite (3.10) as a power series in B, obtaining

$$\zeta(\gamma - \beta - \epsilon)\left\{1 + \sum_{k=1}^\infty B^k \sum_{n=2}^\infty \left[\sum_{\alpha_1+\alpha_2+\cdots=k} C_{\alpha_1\alpha_2\cdots}\right] n^{\beta-\gamma+\epsilon}\right\}.$$

Here the square bracket gives the number $f_k(n)$ of factorizations of n into k factors each greater than 1, order being essential. It is well known[4] that

$$\sum_{n=2}^\infty f_k(n)\, n^{-s} = [\zeta(s) - 1]^k, \qquad \Re(s) > 1.$$

Hence (3.10) is a geometric series in B whose sum reduces to

$$\zeta(\gamma - \beta - \epsilon)\,\{1 - B[\zeta(\gamma - \beta - \epsilon) - 1]\}^{-1},$$

the series being convergent for $\gamma > \beta + \epsilon + \rho(B)$. It follows that the double series in (3.7) is absolutely convergent and may be reduced with the aid of the algorithm (3.5). Hence

$$L(z) = \sum_1^\infty b_n F(nz) \tag{3.11}$$

is the only solution of (3.4) which satisfies condition (2.8). From this point onwards the completeness proof proceeds as above.

[4] See L. Kalmár, Matematikai és Fizikai Lapok, 38 (1931) 1–15, and Acta Litterarum ac Scientiarum, 5 (1931) 95–107 for this "factorisatio numerorum" problem which differs from the divisor problem of Piltz.

It remains only to show that the limit found for γ is the best possible, other things being equal. We choose $\gamma = \beta + \rho(B)$,

$$a_m = -Bm^\beta, \qquad m = 2, 3, \cdots. \tag{3.12}$$

This choice makes all the terms positive in (3.6), i.e., it gives the b's the largest values which they can have when $|a_m| \leqq Bm^\beta$. The homogeneous equation corresponding to (3.4) now reads

$$H(z) - B\sum_2^\infty m^\beta H(mz) = 0, \tag{3.13}$$

and admits of the solution

$$H(z) = z^{-\beta-\rho(B)} = z^{-\gamma} \tag{3.14}$$

which satisfies condition (2.8). It follows that $F(z)$ vanishes identically if $L(z) = H(z)$. Since

$$H(z) = \frac{1}{\Gamma(\gamma)}\int_0^1 t^{z-1}\left(\log\frac{1}{t}\right)^{\gamma-1} dt,$$

we can choose

$$\varphi(t) = \frac{1}{\Gamma(\gamma)}\frac{1}{t}\left(\log\frac{1}{t}\right)^{\gamma-1}, \qquad g(t) = 1. \tag{3.15}$$

This choice satisfies the conditions of §2. The corresponding system $S(\varphi(t), a, \lambda) \subset L_1(0, 1)$, but it is not complete in this space because $g(t) = 1$ is orthogonal to every function of the system. The *deficiency* of the system S, i.e., the number of functions $g(t)$ orthogonal to the set, $g(t) \in L_\infty(0, 1)$, equals the number of linearly independent solutions of (3.13) which satisfy (2.8). This number is at least one, but may be larger.

It is of some interest to consider for a moment the two systems

$$\left(\log\frac{1}{t}\right)^{\rho(1)-1}\cdot\frac{1}{t}\frac{t^n}{1-t^n}, \tag{3.16}$$

$$(n = 1, 2, 3, \cdots)$$

$$\left(\log\frac{1}{t}\right)^{\rho(1)-1}\cdot\frac{1}{t}\frac{t^n - 2t^{2n}}{1-t^n}, \tag{3.17}$$

of which the former is complete in $L_1(0, 1)$ and the latter is not. The reason for this striking difference in behavior between two so closely related sets of functions does not seem a priori obvious.

4. **Generating Dirichlet series.** In the previous discussion we have made essential use of the fact that the set of functions $\{F(nz)\}$ is carried into a subset of itself by the transformation $z \mid mz$, in other words, of the property of the natural numbers of being a multiplicative system. Now the integers are brought into our problem by the power series defining the generating function, and it lies at

hand to try if the method does not extend to irregular power series whose exponents form a multiplicative system. This extension will be given in the present paragraph. For the sake of convenience we work with Dirichlet series rather than with irregular power series which means replacing the interval $(0, 1)$ by $(0, \infty)$. We consider only completeness in $L_1(0, \infty)$, the extension to $L_p(0, \infty)$ and $C[0, \infty]$ being obvious.

Let the positive quantities $\epsilon(n)$ form a multiplicative system, i.e.,

$$\epsilon(mn) = \epsilon(m)\epsilon(n), \qquad m, n = 1, 2, 3, \cdots \tag{4.1}$$

It follows that $\epsilon(1) = 1$. The system is clearly determined by the sub-system $\{\epsilon(p)\}$, where p runs through the primes. We suppose that the abscissa of convergence of the series

$$\sum [\epsilon(p)]^{-s} \tag{4.2}$$

equals σ_0, $0 \leqq \sigma_0 < \infty$. Form the corresponding zeta function

$$\zeta_\epsilon(s) = \prod_p \{1 - [\epsilon(p)]^{-s}\}^{-1} = \sum_{n=1}^{\infty} [\epsilon(n)]^{-s}, \tag{4.3}$$

which is also convergent for $\Re(s) > \sigma_0$, and the adjoint exponential series

$$E_\epsilon(s) = \sum_1^\infty e^{-s\epsilon(n)}. \tag{4.4}$$

For every positive τ, $e^{-x} = O(x^{-\tau})$ as $x \to \infty$. Hence for $\sigma > 0$

$$E_\epsilon(\sigma) = O\left\{\sigma^{-\tau} \sum_{n=1}^{\infty} [\epsilon(n)]^{-\tau}\right\}.$$

The series on the right hand side converges for $\tau > \sigma_0$. Hence (4.4) converges for $\Re(s) > 0$, and

$$E_\epsilon(\sigma) = O(\sigma^{-\sigma_0-\eta}) \tag{4.5}$$

for every positive η. It is frequently permitted to suppress η, but this is of no importance for the following. It is clear that

$$E_\epsilon^\beta(s) = \sum_{n=1}^{\infty} [\epsilon(n)]^\beta e^{-s\epsilon(n)} \tag{4.6}$$

is also convergent for $\Re(s) > 0$, and that

$$E_\epsilon^{(\beta)}(\sigma) = O(\sigma^{-\sigma_0-\Re(\beta)-\eta}) + O(1). \tag{4.7}$$

$\zeta_\epsilon(\sigma)$ is a monotone decreasing function of σ in the interval $\sigma_0 < \sigma < \infty$ which tends to 1 as $\sigma \to \infty$. Put

$$\lim_{\sigma \to \sigma_0 + 0} \zeta_\epsilon(\sigma) = Z.$$

Here Z is either $+\infty$ or a finite quantity >1. The latter possibility cannot be excluded since the set $\{\epsilon(p)\}$ can be chosen arbitrarily, subject only to a con-

vergence condition. Let J_ϵ denote the interval $1 < \sigma \leqq Z$ or $1 < \sigma < \infty$ according as Z is finite or infinite. We shall prove the following

THEOREM 4.1. *Let $\Phi(u)$ be a measurable function which is $\neq 0$ except in a set of measure zero, such that $\Phi(u)\, e^{-\eta u} \in L_1(0, \infty)$ for every $\eta > 0$, and*

$$\int_0^\infty e^{-xu} \mid \Phi(u) \mid du \leqq M_\theta\, x^{-\gamma+\theta}, \qquad x \geqq \delta(\theta) > 0, \tag{4.8}$$

for a fixed $\gamma > 0$ and every $\theta > 0$. Let $a_1 = 1$ and $\mid a_m \mid \leqq B[\epsilon(m)]^\beta$ for a fixed real β. Let $\Re(\lambda_n) \geqq \delta > 0$, $\sum_1^\infty \Re(1/\lambda_n) = +\infty$. Then the system $S(\Phi(u), a, \epsilon, \lambda)$:

$$\Phi(u) \sum_{m=1}^{\infty} a_m\, e^{-u\lambda_n \epsilon(m)}, \qquad n = 1, 2, 3, \cdots, \tag{4.9}$$

is complete in $L_1(0, \infty)$ if $\gamma > \beta + \rho$. Here $\rho = \rho_\epsilon(B) \geqq \sigma_0$, where $\rho_\epsilon(B)$ is the positive root of the equation

$$\zeta_\epsilon(s) = 1 + \frac{1}{B}, \tag{4.10}$$

provided the right hand side of this equation belongs to J_ϵ, otherwise $\rho = \sigma_0$. In the former case there exists a choice of $\Phi(u)$ and of $\{a_m\}$, satisfying the above conditions, such that the corresponding system $S(\Phi(u), a, \epsilon, \lambda)$ is in $L_1(0, \infty)$ without being complete in the space if $\gamma = \beta + \rho_\epsilon(B)$.

PROOF. We proceed as in the previous cases. We have to form

$$F(z) = \sum_{m=1}^{\infty} a_m \int_0^\infty e^{-zu\epsilon(m)}\, \Phi(u)\, G(u)\, du, \qquad \Re(z) > 0. \tag{4.11}$$

The assumptions enable us to show that

$$F(z) = O\left\{x^{-\gamma+\theta} \sum_1^\infty [\epsilon(m)]^{\beta-\gamma+\theta}\right\}. \tag{4.12}$$

The series being convergent, it follows that $F(z)$ is holomorphic in $x > 0$, and, in particular, that $S \subset L_1(0, \infty)$. If the system is not complete, there exists a function $G(u) \in L_\infty(0, \infty)$, not equivalent to zero, such that $F(z) \equiv 0$. Further

$$F(z) = \sum_{m=1}^{\infty} a_m\, L(\epsilon(m)z), \tag{4.13}$$

where

$$L(z) = \int_0^\infty e^{-zu}\, \Phi(u)\, G(u)\, du \tag{4.14}$$

$$\mid L(z) \mid \leqq G\, M_\theta x^{-\gamma+\theta}, \quad x \geqq \delta(\theta) > 0. \tag{4.15}$$

We determine the quantities b_n from the Möbius algorithm (3.5) and form

$$\sum_{n=1}^{\infty} b_n\, F(\epsilon(n)z) = \sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} a_m\, L(\epsilon(mn)z).$$

The absolute convergence of the double series is established as in §3. We obtain a majorant of the form

$$C_\theta\, x^{-\gamma+\theta}\, \zeta_\epsilon(\gamma - \beta - \theta)\, \{1 - B[\zeta_\epsilon(\gamma - \beta - \theta) - 1]\}^{-1}.$$

This is an analytic function of γ which has a right half plane of holomorphism $\Re(\gamma) > \gamma_0 + \theta \geqq \sigma_0 + \theta$, and if $\gamma_0 > \sigma_0$, then $\gamma_0 = \rho_\epsilon(B)$. Hence the double series is absolutely convergent for $\gamma > \gamma_0$, and for such values of γ,

$$L(z) = \sum_{n=1}^{\infty} b_n\, F(\epsilon(n)z) \tag{4.16}$$

is the only solution of (4.13) which satisfies (4.15). In particular, if $F(z) \equiv 0$ so is $L(z)$, whence it follows that $G(u) \sim 0$ and that S is complete.

The construction of a 'Gegenbeispiel' follows the same lines as in §3. We take

$$a_m = -B[\epsilon(m)]^\beta, \qquad m = 2, 3, \cdots. \tag{4.17}$$

Then the homogeneous equation

$$H(z) - B \sum_{m=2}^{\infty} [\epsilon(m)]^\beta\, H(\epsilon(m)z) = 0 \tag{4.18}$$

is satisfied by

$$H(z) = z^{-\beta-\rho_\epsilon(B)} = z^{-\gamma}, \tag{4.19}$$

provided $1 + 1/B$ belongs to J_ϵ and we assume $\gamma = \beta + \rho_\epsilon(B)$. Taking

$$\Phi(u) = \frac{1}{\Gamma(\gamma)}\, u^{\gamma-1}, \qquad G(u) = 1, \tag{4.20}$$

we obtain a system $S(\Phi(u), a, \epsilon, \lambda) \subset L_1(0, \infty)$, but not complete in this space since $G(u) = 1$ is orthogonal to every function of the set. If $1 + 1/B$ does not belong to J_ϵ, this construction breaks down. Moreover, a glance at the proof shows that the completeness of the system S is preserved for $\gamma = \beta + \sigma_0$. On the other hand, we cannot permit any smaller value for γ, because, as is easily shown by examples, S will then cease to belong to $L_1(0, \infty)$, so that the completeness problem becomes meaningless.

Theorem 4.1 is a generalization of Theorem 3. We can find a corresponding generalization of Theorem 2 in which we merely replace the exponents $\{m\}$ by another multiplicative system $\{\epsilon(m)\}$ and leave the coefficients $\{m^\beta\}$ alone. The analysis will then involve the special inversion formula again. This, however, is due essentially to the fact that $\{m^\beta\}$ is a multiplicative system. We are thus led to a further generalization of Theorem 2 in which both the exponents $\{m\}$ and the coefficients $\{m^\beta\}$ are replaced by multiplicative systems, $\{\epsilon(m)\}$ and $\{\alpha(m)\}$ respectively.

We make the same assumptions on $\{\epsilon(m)\}$ as above. Let $\alpha(m)$ be real or complex, $\alpha(1) = 1$,

$$\alpha(mn) = \alpha(m)\alpha(n), \qquad m, n = 1, 2, 3, \cdots, \tag{4.21}$$

and let the abscissa of absolute convergence of the series

$$\sum_{(p)} \alpha(p)\,[\epsilon(p)]^{-s} \tag{4.22}$$

equal τ_0, $0 \leqq \tau_0 < \infty$. Then the series

$$\sum_{n=1}^{\infty} \alpha(n)\, e^{-s\epsilon(n)}$$

is absolutely convergent for $\Re(s) > 0$. Let $S(\Phi(u), \alpha, \epsilon, \lambda)$ denote the system of functions

$$\Phi(u) \sum_{m=1}^{\infty} \alpha(m)\, e^{-u\lambda_n \epsilon(m)}, \qquad n = 1, 2, 3, \cdots. \tag{4.23}$$

THEOREM 4.2. *Let $\Phi(u)$ and $\{\lambda_n\}$ satisfy the assumptions of Theorem 4.1. The set $S(\Phi(u), \alpha, \epsilon, \lambda)$ is complete in $L_1(0, \infty)$ provided $\gamma > \tau_0$.*

PROOF. As usual we form

$$F(z) = \sum_{m=1}^{\infty} \alpha(m) \int_0^{\infty} e^{-\epsilon(m)zu}\, \Phi(u)\, G(u)\, du, \qquad \Re(z) > 0. \tag{4.24}$$

Then

$$F(z) = O\left\{x^{-\gamma+\theta} \sum_{m=1}^{\infty} |\, \alpha(m)\,|\, [\epsilon(m)]^{-\gamma+\theta}\right\}. \tag{4.25}$$

The series is convergent for sufficiently small $\theta > 0$. If the system S is not complete, there exists a $G(u) \in L_\infty(0, \infty)$, not equivalent to zero, such that $F(z) \equiv 0$. But, with the usual notation,

$$F(z) = \sum_{m=1}^{\infty} \alpha(m)\, L(\epsilon(m)z),$$

and

$$\sum_{n=1}^{\infty} \mu(n)\,\alpha(n)\, F(\epsilon(n)z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(n)\,\alpha(mn)\, L(\epsilon(mn)z)$$

$$= \sum_{k=1}^{\infty} \alpha(k)\, L(\epsilon(k)z) \sum_{d \mid k} \mu(d) = L(z),$$

the double series being absolutely convergent. Hence $L(z) \equiv 0$, $G(t) \sim 0$, and S is complete.

It is clear that in Theorem 4.1 we can replace the assumption $|\, a_m\,| \leqq B[\epsilon(m)]^{\beta}$ by $|\, a_m\,| \leqq B\alpha(m)$, where the $\{\alpha(m)\}$ form a multiplicative system, if the other assumptions are suitably modified.

5. **Completeness in $C[0, 1]$.** In this paragraph we shall investigate the question of the completeness of the system $1 + S(\beta + 1, \beta, \lambda)$ in $C[0, 1]$ which was raised in our first note.

THEOREM 5. *The system* $1 + S(\beta + 1, \beta, \lambda)$ *is complete in*

$$C[0, 1] \text{ if } -1 < \beta \leqq 2,$$

and

$$\Re(\lambda_n) \geqq \delta > 0, \qquad \sum_1^\infty \Re(1/\lambda_n) = +\infty.$$

For the proof we need some lemmas.

LEMMA 5.1. $Q_\beta(t) \equiv t^{-1}(1 - t)^{\beta+1} k_\beta(t)$ *is an increasing function of* t *in* $0 < t < 1$ *for* $1 \leqq \beta$.

PROOF. A straight forward calculation shows that the derivative of $Q_\beta(t)$ equals

$$(1 - t)^\beta \sum_{n=0}^\infty (n + 1)\,[(n + 2)^\beta - (n + 1)^{\beta-1}(n + 1 + \beta)]\, t^n > 0$$

since every square bracket is positive.[5]

LEMMA 5.2. *For* $0 < t < 1$, $-1 < \beta \leqq 2$, $n = 1, 2, 3, \cdots$

$$\delta(t, \beta, n) \equiv \frac{n^{\beta+1} t^n}{(1 - t^n)^{\beta+1}} - \frac{(n + 1)^{\beta+1} t^{n+1}}{(1 - t^{n+1})^{\beta+1}} > 0.$$

PROOF. We begin with the case $\beta = 2$. A simple calculation shows that $\delta(t, 2, n)(t - 1)^2$ is holomorphic at $t = 1$ and tends to a positive limit as $t \to 1$. Writing $P_n(t)$ for the product of $\delta(t, 2, n)$ with $t^{-n}(1 - t^n)^3(1 - t^{n+1})^3$ we get

$$P_n(t) = A - Bt + 3(B - A)t^{n+1} - 3Bt^{2n+1} + 3At^{2n+2} + Bt^{3n+1} - At^{3n+3},$$

where $A = n^3$, $B = (n + 1)^3$.

By the rule of Descartes, the equation $P_n(t) = 0$ has at most five positive roots. But $P_n(t) = (1 - t)^4 Q_n(t)$, where $Q_n(1) > 0$, further $P_n(0) > 0$ and $P_n(t) \to -\infty$ as $t \to +\infty$. Hence there are actually five roots, but the fifth root must be greater than 1, so that the lemma is true for $\beta = 2$. Hence for $-1 < \beta < 2$

$$\frac{n}{1 - t^n} > \frac{(n + 1)t^{\frac{1}{3}}}{1 - t^{n+1}} > \frac{(n + 1)t^{\frac{1}{\beta+1}}}{1 - t^{n+1}},$$

which proves the lemma.[6]

LEMMA 5.3. *For* $0 < t < 1$, $1 \leqq \beta \leqq 2$, $n = 1, 2, 3, \cdots$

$$\Delta(t, \beta, n) \equiv n^{\beta+1} k_\beta(t^n) - (n + 1)^{\beta+1} k_\beta(t^{n+1}) > 0.$$

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[5] See Hardy, Littlewood, and Pólya, *Inequalities*, p. 39, formula (2.15.1).

[6] For the special cases $\beta = 0$ and 1 see Hardy, Littlewood, and Pólya, *op. cit.*, p. 41, formula (2.15.6), and p. 99, Theorem 131, resp.

PROOF. Since $t^{n+1} < t^n$, Lemma 5.1 implies

$$\Delta(t, \beta, n) = \frac{n^{\beta+1} t^n}{(1 - t^n)^{\beta+1}} t^{-n}(1 - t^n)^{\beta+1} k_\beta(t^n)$$

$$- \frac{(n+1)^{\beta+1} t^{n+1}}{(1 - t^{n+1})^{\beta+1}} t^{-n-1}(1 - t^{n+1})^{\beta+1} k_\beta(t^{n+1})$$

$$> t^{-n-1}(1 - t^{n+1})^{\beta+1} k_\beta(t^{n+1}) \delta(t, \beta, n) > 0$$

by Lemma 5.2. It is of some interest to note that $\Delta(t, \beta, n)$ is holomorphic at $t = 1$ when β is not a negative integer, and

$$\Delta(1, \beta, n) = \zeta(-\beta)[n^{\beta+1} - (n+1)^{\beta+1}].$$

This is readily obtained from the known representation of $k_\beta(w)$ near $w = 1$.[7] It shows that Lemma 5.3 is not true for $4k + 2 < \beta < 4k + 4$, $k = 0, 1, 2, \cdots$, and we do not know if it is true for any value of β outside of the range $1 \leqq \beta \leqq 2$.

LEMMA 5.4. *For* $-1 < \beta \leqq 2$ *the series*

$$\sum_{n=1}^{\infty} \mu(n) n^\beta k_\beta(t^n)(1 - t)^{\beta+1} = t(1 - t)^{\beta+1}$$

converges boundedly[8] *in* $0 < t < 1$.

PROOF. The series is obviously absolutely convergent for $|t| < 1$, and the sum of the series is found by applying the inversion formula of Möbius to the power series of $k_\beta(t)$. In order to prove the bounded convergence we apply summation by parts, and note that

$$|\bar{\mu}(n)| \leqq 1 \text{ where } \bar{\mu}(n) = \sum_{m=1}^{n} \frac{\mu(m)}{m}, \tag{5.1}$$

the proof being elementary.[9] Hence

$$S_{m,\beta}(t) \equiv \sum_{n=1}^{m} \mu(n) n^\beta k_\beta(t^n) = \sum_{n=1}^{m-1} \bar{\mu}(n)[n^{\beta+1} k_\beta(t^n) - (n+1)^{\beta+1} k_\beta(t^{n+1})]$$

$$+ \bar{\mu}(m) m^{\beta+1} k_\beta(t^m).$$

If $1 \leqq \beta \leqq 2$, Lemma 5.3 applies and shows that all square brackets are positive. Hence by (5.1) $S_{m,\beta}(t)$ is dominated by the expression obtained by replacing $\bar{\mu}(n)$ throughout by unity, i.e., by $k_\beta(t)$. Since $(1 - t)^{\beta+1} k_\beta(t)$ is bounded in $0 < t < 1$, the lemma is proved for the case $1 \leqq \beta \leqq 2$. But we can reduce the remaining cases to this one. Let us first assume $0 \leqq \beta < 1$. Since

$$k_\beta(t) = \int_0^1 u^{-1} k_{\beta+1}(tu)\, du, \tag{5.2}$$

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[7] Cf. E. Lindelöf, *Calcul des residues*, p. 139.

[8] More than this is true. We could actually prove uniform convergence, but the proof seems to require the prime number theorem or results of similar depth.

[9] See E. Landau, *Vorlesungen über Zahlentheorie*, I, p. 21.

we have

$$S_{m,\beta}(t) = \sum_{n=1}^{m} \mu(n) n^{\beta} \int_0^1 u^{-1} k_{\beta+1}(t^n u)\, du$$

$$= \sum_{n=1}^{m} \mu(n) n^{\beta+1} \int_0^1 \tau^{-1} k_{\beta+1}(t^n \tau^n)\, d\tau$$

$$= \int_0^1 \tau^{-1} S_{m,\beta+1}(t\tau)\, d\tau.$$

But $| S_{m,\beta+1}(t\tau) | \leqq k_{\beta+1}(t\tau)$ since $1 \leqq \beta + 1 \leqq 2$. Then (5.2) shows that $S_{m,\beta}(t)$ is still dominated by $k_{\beta}(t)$, so that the lemma is also true for $0 \leqq \beta \leqq 1$, and the argument can evidently be repeated for the interval $-1 < \beta < 0$.

PROOF OF THEOREM 5. If the system $1 + S(\beta + 1, \beta, \lambda)$ should not be complete in $C[0, 1]$, there exists a function $g(t)$ of bounded variation in $0 \leqq t \leqq 1$ such that $g(0) = g(1)$ and

$$\int_0^1 k_{\beta}(t^{\lambda_n})(1 - t)^{\beta+1}\, dg(t) = 0, \qquad n = 1, 2, 3, \cdots.$$

As usual we conclude that

$$F(z) = \int_0^1 k_{\beta}(t^z)(1 - t)^{\beta+1}\, dg(t) \equiv 0. \tag{5.3}$$

Form

$$L(z) = \int_0^1 t^z (1 - t)^{\beta+1}\, dg(t), \qquad \Re(z) = x > 0. \tag{5.4}$$

By Lemma 5.4

$$t^x = \sum_{n=1}^{\infty} \mu(n) n^{\beta} k_{\beta}(t^{nx}), \tag{5.5}$$

where the series converges boundedly after multiplication by $(1 - t^x)^{\beta+1}$. Since $[(1 - t)/(1 - t^x)]^{\beta+1}$ is a bounded function of t in $0 < t < 1$ for a fixed $x > 0$, the bounded convergence implies that

$$L(x) = \lim_{m \to \infty} \int_0^1 \left\{ \sum_{n=1}^{m} \mu(n) n^{\beta} k_{\beta}(t^{nx}) \right\} (1 - t)^{\beta+1}\, dg(t)$$

$$= \sum_{n=1}^{\infty} \mu(n) n^{\beta} F(nx) \equiv 0.$$

$L(z)$ being analytic, we have $L(z) \equiv 0$, whence we conclude that $g(t)$ is constant as in our first note.

In particular, the systems

$$1, \frac{(1 - t)t^n}{1 - t^n} \text{ and } 1, \frac{(1 - t)^2 t^n}{(1 - t^n)^2}$$

are complete in $C[0, 1]$. This settles the problem raised at the end of our first note. It is quite likely that Theorem 5 holds over a greater range of values of β than $-1 < \beta \leqq 2$, but our present method seems to be limited to this interval.

Our two notes on completeness of Lambert functions raise a number of problems. There are applications to the Tauberian theory of various classes of Lambert series, and possibly also to the prime number theorem. The character and range of validity of the inversion formula of Möbius, its number-theoretical features, and its possible extension to other inversion problems invite investigation. The authors plan to return to these problems individually or jointly, in later papers.

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A CONVERSE OF POISSON'S THEOREM IN THE THEORY OF PROBABILITY

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I

Dr. S. M. Ulam communicated the following problem to the present author.

Suppose there are given N urns each containing the same total number of white and black balls; let $p_\nu (0 \leqq p_\nu \leqq 1, \nu = 1, \cdots, N)$ denote the unknown ratio of white balls to all balls in the ν^{th} urn. We assume that in each urn all values $0 \leqq p_\nu \leqq 1$ are equi-probable a priori, an that there is no dependence between the distributions of white and black balls in the different urns.

Let us assume that on drawing one ball from every urn, white appears m times and black n times, $m + n = N$. On the strength of this assumption it is easy to give an expression for the function $P_{m,n}(p)$ representing the probability for the relation

$$0 \leqq \frac{p_1 + \cdots + p_N}{N} \leqq p, \qquad 0 < p < 1. \tag{1}$$

In fact, since $p_\nu dp_\nu$ is the infinitesimal probability for drawing a white ball from the ν^{th} urn, and $(1 - p_\nu)dp_\nu$ for drawing a black one we obtain, by the theorem of Bayes, the relation

$$P_{m,n}(p) = \frac{J_{m,n}(p)}{J_{m,n}(1)}, \tag{2}$$

where

$$J_{m,n}(p) = \int_{A_{p,N}} p_1 \cdots p_m (1 - p_{m+1}) \cdots (1 - p_N) dp_1 \cdots dp_N, \tag{3}$$

the point set $A_{p,N}$ being the intersection of the unit cube

$$0 \leqq p_\nu \leqq 1, \qquad \nu = 1, \cdots, N \tag{4}$$

with the layer (1). The question arises as to whether the function $P_{m,n}(p)$ has any distinctive properties as N tends to ∞. Dr. Ulam conjectured that for $m = n$, or, more generally,

$$\lim_{N \to \infty} \frac{m}{N} = \frac{1}{2}, \tag{5}$$

the function $P_{m,n}(p)$ will satisfy the limit relation

$$\lim_{N \to \infty} P_{m,n}(p) = \begin{cases} 0, \text{ if } p < \frac{1}{2} \\ 1, \text{ if } p > \frac{1}{2} \end{cases}. \tag{6}$$

The interpretation would be as follows. If, for a large number of urns, on drawing one ball from each urn, white appears nearly as often as black then it is practically certain that the average number of white balls in all urns taken together is nearly equal to the corresponding average of black balls. We shall see that this conjecture is correct.

The problem can be generalized. We replace condition (5) by the more general condition

$$\lim_{N\to\infty} \frac{m}{N} = t, \qquad 0 \leqq t \leqq 1 \tag{7}$$

(t being the "tested" relative average of white balls). Then one might expect the relation

$$\lim_{N\to\infty} P_{m,n}(p) = \begin{cases} 0, & 0 \leqq p < p_0 \\ 1, & p_0 < p \leqq 1 \end{cases} \tag{8}$$

with

$$p_0 = t. \tag{9}$$

We shall see that (8) is true but not (9). Relation (9) is to be replaced by

$$p_0 = \tfrac{1}{3}(t+1), \tag{10}$$

in other words there does exist an asymptotic "probable" relative average p_0 but its relation to the tested average t is not so trivial as one might at first expect after treating the special cast $t = \frac{1}{2}$. In order to explain relation (10) we shall generalize our original assumption that all values p_ν in the ν^{th} urn are equi-probable a priori. We shall assume that the probability for p_ν to be contained in the interval $(p_\nu, p_\nu + dp_\nu)$ is given by the "differential"

$$df(p_\nu), \qquad \left[\int_0^1 df(p) = 1\right]. \tag{11}$$

We shall require some conditions concerning the moments

$$b_k = \int_0^1 p^k df(p),$$

namely

$$0 < b_2 < b_1 < 1, \tag{12}$$

$$b_2^2 - b_1 b_3 < 0, \tag{13}$$

$$(b_1 - b_2)^2 - (1 - b_1)(b_2 - b_3) < 0.^1 \tag{14}$$

Formula (3) is to be replaced by the more general formula

$$J_{m,n}(p) = \int_{A_{p,N}} p_1 \cdots p_m \cdot (1 - p_{m+1}) \cdots (1 - p_N) df(p_1) \cdots df(p_N). \tag{15}$$

[1] It can be shown that these conditions are certainly fulfilled if the probability $df(p)$ is not concentrated in the two end points $p = 0, 1$ or in one intermediate point exclusively.

Relation (8) is again true and (10) is to be replaced by

$$p_0 = \rho t + \sigma(1 - t), \tag{16}$$

where

$$\rho = \frac{b_2}{b_1}, \qquad \sigma = \frac{b_1 - b_2}{1 - b_1}. \tag{17}$$

Relation (16) is the same as

$$p_0 = b_1 + \frac{b_2 - b_1^2}{b_1(1 - b_1)}(t - b_1). \tag{18}$$

The number b_1 is the expected value of p with respect to the a priori probability $df(p)$. We may also claim it to be the relative average of white balls that one should expect a priori. As such we denote it by e_0. The number $\delta = b_2 - b_1^2$ is the dispersion of the a priori distribution $df(p)$. Hence we may write

$$p_0 = e_0 + \frac{\delta}{e_0(1 - e_0)}(t - e_0).$$

In other words, the probable relative average p_0 is equal to the a priori average e_0 plus a certain multiple of the excess of the tested average t over the expected average e_0. Also, the factor $\delta/[e_0(1 - e_0)]$ lies between 0 and 1. Thus we finally obtain the following statement: the probable average p_0 is a weighted mean between the a priori average e_0 and the tested average t, the values of the weights depending only on the expectation and dispersion of the a priori probability $df(p)$.

II

If $\Phi(p_1, \cdots, p_N)$ is defined and absolutely integrable over the entire Euclidean space of the variables p_ν, and $D(\xi)$ is 1 in the interval $\eta \leqq \xi \leqq \zeta$ and 0 outside, then the integral

$$\int_{-\infty}^{\infty} \cdots \int \Phi(p_1, \cdots, p_N) D(p_1 + \cdots + p_N) dp_1 \cdots dp_N$$

may be written in the form

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} E(\alpha) \frac{e^{-i\eta\alpha} - e^{-i\zeta\alpha}}{i\alpha} d\alpha$$

where

$$E(\alpha) = \int_{-\infty}^{\infty} \cdots \int \Phi(p_1, \cdots, p_N) e^{i\alpha(p_1 + \cdots + p_N)} dp_1 \cdots dp_N .^2$$

Substituting for $\Phi(p_1, \cdots, p_N)$ the expression

$$p_1 \cdots p_m(1 - p_{m+1}) \cdots (1 - p_N) \frac{df(p_1)}{dp_1} \cdots \frac{df(p_N)}{dp_N} \tag{19}$$

[2] See S. Bochner, *Vorlesungen über Fouriersche Integrale*, (1932), s. 61, Satz 16.

in the unit cube $(4)^3$ and 0 outside, and putting $\eta = -Np$, $\zeta = Np$, we obtain for the function (15) the identity

$$J_{m,n}(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi_0(\alpha)^m \psi_0(\alpha)^n \frac{\sin Np\alpha}{\alpha}\, d\alpha, \tag{20}$$

where

$$\varphi_0(\alpha) = \int_0^1 pe^{i\alpha p}\, df(p), \qquad \psi_0(\alpha) = \int_0^1 (1-p)\, e^{i\alpha p}\, df(p).$$

Obviously

$$\varphi_0(\alpha) = b_1 + \frac{b_2}{1!}(i\alpha) + \frac{b_3}{2!}(i\alpha)^2 + \cdots$$

$$\psi_0(\alpha) = 1 - b_1 + \frac{b_1 - b_2}{1!}(i\alpha) + \frac{b_2 - b_3}{2!}(i\alpha)^2 + \cdots$$

and

$$J_{m,n}(1) = b_1^m(1-b_1)^n.$$

Hence introducing the functions

$$\varphi(\alpha) = 1 + i\frac{b_2}{b_1}\alpha - \frac{b_3}{2b_1}\alpha^2 + \cdots$$

$$\psi(\alpha) = 1 + i\frac{b_1 - b_2}{1-b_1}\alpha - \frac{b_2 - b_3}{2(1-b_1)}\alpha^2 + \cdots$$

we obtain

$$P_{m,n}(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi(\alpha)^m \psi(\alpha)^n \frac{\sin Np\alpha}{\alpha}\, d\alpha. \tag{21}$$

Since we only know

$$|\varphi(\alpha)| \leqq 1, \qquad |\psi(\alpha)| \leqq 1,$$

this integral may be absolutely divergent at $\alpha = \pm\infty$. This complication will be avoided if we consider the function

$$Q_{m,n}(p) = \int_0^p P_{m,n}(p)\, dp. \tag{22}$$

It has the value

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \varphi(\alpha)^m \psi(\alpha)^n \frac{(\sin \frac{1}{2}Np\alpha)^2}{N\alpha^2}\, d\alpha. \tag{23}$$

Using the abbreviations (17) we now put

$$\varphi(\alpha) = e^{ip\alpha}\varphi_1(\alpha), \qquad \psi(\alpha) = e^{i\sigma\alpha}\psi_1(\alpha).$$

[3] The argument loc. cit. can be easily extended so as to cover an expression of the type (19), for an everywhere dense set of values η, ζ.

In the neighborhood of $\alpha = 0$ we have

$$\varphi_1(\alpha) = 1 - A\alpha^2 + \cdots, \qquad \psi_1(\alpha) = 1 - B\alpha^2 + \cdots; \tag{24}$$

and it follows from (13) and (14) respectively that $A > 0$, $B > 0$. Hence there exists an $\alpha_0 > 0$ such that for all $-\alpha_0 < \alpha < \alpha_0$ and for all m, n

$$| \varphi_1(\alpha)^m \psi_1(\alpha)^n - e^{-(Am+Bn)\alpha^2} | \leqq C \cdot N\alpha^3 e^{-D \cdot N\alpha^2}, \tag{25}$$

where C and D denote suitable constants. Therefore $Q_{m,n}(p)$ differs from

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \exp\left[-(Am + Bn)\alpha^2 + i(\rho m + \sigma n)\alpha\right] \frac{(\sin \frac{1}{2} Np\alpha)^2}{N\alpha^2} \, d\alpha \tag{26}$$

by

$$O\left(\int_{-\alpha_0}^{\alpha_0} \alpha \, e^{-D \cdot N\alpha^2} \, d\alpha\right) + O\left(\int_{\alpha_0}^{\infty} \frac{d\alpha}{N\alpha^2}\right) = O(N^{-1}).$$

The positive numbers

$$\lambda = \frac{Am + nB}{m + n}, \qquad \mu = \frac{\rho m + \sigma n}{m + n} \tag{27}$$

depend on m, n, but owing to our assumptions (12)-(14) they lie between positive bounds. Replacing α by α/N, (26) may be written

$$\frac{2}{\pi} \int_{-\infty}^{\infty} e^{-\frac{\lambda\alpha^2}{N} + i\mu\alpha} \frac{(\sin \frac{1}{2} p\alpha)^2}{\alpha^2} \, d\alpha. \tag{28}$$

This differs from

$$\frac{2}{\pi} \int_{-\infty}^{\infty} e^{i\mu\alpha} \frac{(\sin \frac{1}{2} p\alpha)^2}{\alpha^2} \, d\alpha \tag{29}$$

by a term which is majorized by

$$\int_0^1 \left(1 - e^{-\frac{\alpha^2}{N}}\right) d\alpha + \int_1^{\infty} \left(1 - e^{-\frac{\alpha^2}{N}}\right) \frac{d\alpha}{\alpha^2} = O(N^{-1}) + O(N^{-\frac{1}{2}}).$$

The integral (29) can easily be evaluated.[4] It has the value O for $p < \mu$ and $p - \mu$ for $p > \mu$. Hence, altogether,

$$\int_0^p P_{m,n}(p) \, dp = \begin{cases} O(N^{-\frac{1}{2}}) & \text{if } p < \mu \\ p - \mu + O(N^{-\frac{1}{2}}), & \text{if } p > \mu; \end{cases} \tag{30}$$

the error estimation $O(N^{-\frac{1}{2}})$ appearing in this formula is valid uniformly in $O \leqq p \leqq 1$, and uniformly in all m, n for which $m + n = N$. Since $P_{m,n}(p)$ is monotonely non-decreasing, (30) allows the conclusion

$$\begin{aligned} P_{m,n}(\mu - g) &= O(N^{-\frac{1}{2}} g^{-1}) \\ P_{m,n}(\mu + g) &= 1 + O(N^{-\frac{1}{2}} g^{-1}), \end{aligned} \tag{31}$$

if $g > 0$.

[4] Loc. cit., p. 15, formula (14).

Assuming now the existence of the limit (7), the quantity μ will converge towards the number p_0 that was defined by (16). And we may conclude from (31) that for each $\epsilon > 0$

$$P_{m,n}(p_0 - \epsilon) = O(N^{-\frac{1}{2}})$$
$$P_{m,n}(p_0 + \epsilon) = 1 + O(N^{-\frac{1}{2}}). \tag{32}$$

The estimate (30) can be strengthened. The expression (28) is the integral from 0 to p of the function

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\frac{\lambda\alpha^2}{N}+i\mu\alpha}\,\frac{\sin p\alpha}{\alpha}\,d\alpha \tag{33}$$

which has the value

$$\frac{\sqrt{N}}{2\sqrt{(\pi\lambda)}}\int_{-p}^{p} e^{-\frac{N(\mu+\xi)^2}{4\lambda}}\,d\xi = \frac{\sqrt{N}}{2\sqrt{(\pi\lambda)}}\int_{-\infty}^{p-\mu} e^{-\frac{N\xi^2}{4\lambda}}\,d\xi + O(N^{-1}). \tag{34}$$

Hence, in terms of the error function

$$\psi(x) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{x} e^{-\frac{x^2}{4}}\,dx \tag{35}$$

we obtain the relation

$$\int_0^p P_{m,n}(p)dp = \sqrt{(\lambda/N)}\int_{-\infty}^{(p-\mu)\sqrt{(N/\lambda)}} \psi(x)\,dx + O(N^{-1}) \tag{36}$$

or

$$\int_0^p P_{m,n}(p)dp = ((p-\mu)\,\psi\,((p-\mu)\sqrt{(N/\lambda)})$$
$$+ \sqrt{(\lambda/N\pi)}\,e^{-\frac{N(p-\mu)^2}{4\lambda}} + O(N^{-1}). \tag{37}$$

This relation can be easily seen to include (30). For $p = \mu$, (37) becomes

$$\int_0^\mu P_{m,n}(p)dp = \sqrt{(\lambda/N\pi)} + O(N^{-1}). \tag{38}$$

III

In order to obtain a relation similar to (37) for the function $P_{m,n}(p)$ itself we make more stringent assumptions concerning the nature of the a priori distribution $df(p)$. We require that for every $\alpha_0 > O$,

$$\int_{-\infty}^{-\alpha_0} + \int_{\alpha_0}^{\infty} |\,\varphi(\alpha)^m\,\psi(\alpha)^n\,|\,\frac{d\alpha}{\alpha} = O(N^{-\frac{1}{2}}). \tag{39}$$

This will be amply satisfied if the distribution $df(p)$ arises from a density, $df(p) = g(p)dp$, and the density $g(p)$ is of bounded variation. Using again (25) we find that $P_{m,n}(p)$ differs from (33) by $O(N^{-\frac{1}{2}})$. Hence we obtain

$$P_{m,n}(p) = \psi((p-\mu)\sqrt{(N/\lambda)}) + O(N^{-\frac{1}{2}}). \tag{40}$$

In particular, uniformly in all positive and negative values g,

$$P_{m,n}\left(\mu + \frac{g}{\sqrt{N}}\right) = \frac{1}{2\sqrt{(\pi\lambda)}} \int_{-\infty}^{g} e^{-\frac{\xi^2}{4\lambda}} d\xi + O(N^{-\frac{1}{2}}). \tag{41}$$

We now introduce the assumption

$$\frac{m}{N} = t + O(N^{-1})$$

which is more stringent than (7). Obviously $\mu = p_0 + O(N^{-1})$, and $\lambda = \lambda_0 + O(N^{-1})$ where

$$\lambda_0 = At + B(1 - t).$$

It is easily seen that (33) differs only by $O(N^{-\frac{1}{2}})$ from the same integral in which λ, μ are replaced by λ_0, p_0. Therefore, (40) and (41) may be replaced by

$$P_{m,n}(p) = \psi((p - p_0)\sqrt{(N/\lambda_0)}) + O(N^{-\frac{1}{2}}) \tag{42}$$

$$P_{m,n}\left(p_0 + \frac{g}{\sqrt{N}}\right) = \frac{1}{2\sqrt{\pi\lambda_0}} \int_{-\infty}^{g} e^{-\frac{\xi^2}{4\lambda_0}} d\xi + O(N^{-\frac{1}{2}}). \tag{43}$$

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THE LOGIC OF QUANTUM MECHANICS

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1. **Introduction.** One of the aspects of quantum theory which has attracted the most general attention, is the novelty of the logical notions which it presupposes. It asserts that even a complete mathematical description of a physical system $\mathfrak{S}$ does not in general enable one to predict with certainty the result of an experiment on $\mathfrak{S}$, and that in particular one can never predict with certainty both the position and the momentum of $\mathfrak{S}$ (Heisenberg's Uncertainty Principle). It further asserts that most pairs of observations are incompatible, and cannot be made on $\mathfrak{S}$ simultaneously (Principle of Non-commutativity of Observations).

The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic. Our main conclusion, based on admittedly heuristic arguments, is that one can reasonably expect to find a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces with respect to *set products*, *linear sums*, and *orthogonal complements*—and resembles the usual calculus of propositions with respect to *and*, *or*, and *not*.

In order to avoid being committed to quantum theory in its present form, we have first (in §§2–6) stated the heuristic arguments which suggest that such a calculus is the proper one in quantum mechanics, and then (in §§7–14) reconstructed this calculus from the axiomatic standpoint. In both parts an attempt has been made to clarify the discussion by continual comparison with classical mechanics and its propositional calculi. The paper ends with a few tentative conclusions which may be drawn from the material just summarized.

I. PHYSICAL BACKGROUND

2. **Observations on physical systems.** The concept of a physically observable "physical system" is present in all branches of physics, and we shall assume it.

It is clear that an "observation" of a physical system $\mathfrak{S}$ can be described generally as a writing down of the readings from various[1] compatible measurements. Thus if the measurements are denoted by the symbols $\mu_1, \cdots, \mu_n$, then

[1] If one prefers, one may regard a set of compatible measurements as a single composite "measurement"—and also admit non-numerical readings—without interfering with subsequent arguments.

Among conspicuous observables in quantum theory are position, momentum, energy, and (non-numerical) symmetry.

an observation of $\mathfrak{S}$ amounts to specifying numbers $x_1, \cdots, x_n$ corresponding to the different μ_k.

It follows that the most general form of a prediction concerning $\mathfrak{S}$ is that the point $(x_1, \cdots, x_n)$ determined by actually measuring $\mu_1, \cdots, \mu_n$, will lie in a subset S of $(x_1, \cdots, x_n)$-space. Hence if we call the $(x_1, \cdots, x_n)$-spaces associated with $\mathfrak{S}$, its "observation-spaces," we may call the subsets of the observation-spaces associated with any physical system $\mathfrak{S}$, the "experimental propositions" concerning $\mathfrak{S}$.

3. **Phase-spaces.** There is one concept which quantum theory shares alike with classical mechanics and classical electrodynamics. This is the concept of a mathematical "phase-space."

According to this concept, any physical system $\mathfrak{S}$ is at each instantly hypothetically associated with a "point" p in a fixed phase-space Σ; this point is supposed to represent mathematically the "state" of $\mathfrak{S}$, and the "state" of $\mathfrak{S}$ is supposed to be ascertainable by "maximal"[2] observations.

Furthermore, the point p_0 associated with $\mathfrak{S}$ at a time t_0, together with a prescribed mathematical "law of propagation," fix the point p_t associated with $\mathfrak{S}$ at any later time t; this assumption evidently embodies the principle of *mathematical causation.*[3]

Thus in classical mechanics, each point of Σ corresponds to a choice of n position and n conjugate momentum coördinates—and the law of propagation may be Newton's inverse-square law of attraction. Hence in this case Σ is a region of ordinary $2n$-dimensional space. In electrodynamics, the points of Σ can only be specified after certain *functions*—such as the electromagnetic and electrostatic potential—are known; hence Σ is a function-space of infinitely many dimensions. Similarly, in quantum theory the points of Σ correspond to so-called "wave-functions," and hence Σ is again a function-space—usually[4] assumed to be Hilbert space.

In electrodynamics, the law of propagation is contained in Maxwell's equations, and in quantum theory, in equations due to Schrödinger. In any case, the law of propagation may be imagined as inducing a steady fluid motion in the phase-space.

It has proved to be a fruitful observation that in many important cases of classical dynamics, this flow conserves volumes. It may be noted that in quantum mechanics, the flow conserves distances (i.e., the equations are "unitary").

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[2] L. Pauling and E. B. Wilson, *"An introduction to quantum mechanics,"* McGraw-Hill, 1935, p. 422. Dirac, *"Quantum mechanics,"* Oxford, 1930, §4.

[3] For the existence of mathematical causation, cf. also p. 65 of Heisenberg's *"The physical principles of the quantum theory,"* Chicago, 1929.

[4] Cf. J. von Neumann, *"Mathematische Grundlagen der Quanten-mechanik,"* Berlin, 1931. p. 18.

4. **Propositions as subsets of phase-space.** Now before a phase-space can become imbued with reality, its elements and subsets must be correlated in some way with "experimental propositions" (which are subsets of different observation-spaces). Moreover, this must be so done that set-theoretical inclusion (which is the analogue of logical implication) is preserved.

There is an obvious way to do this in dynamical systems of the classical type.[5] One can measure position and its first time-derivative velocity—and hence momentum—explicitly, and so establish a one-one correspondence which preserves inclusion between subsets of phase-space and subsets of a suitable observation-space.

In the cases of the kinetic theory of gases and of electromagnetic waves no such simple procedure is possible, but it was imagined for a long time that "demons" of small enough size could by tracing the motion of each particle, or by a dynamometer and infinitesimal point-charges and magnets, measure quantities corresponding to every coördinate of the phase-space involved.

In quantum theory not even this is imagined, and the possibility of predicting in general the readings from measurements on a physical system $\mathfrak{S}$ from a knowledge of its "state" is denied; only statistical predictions are always possible.

This has been interpreted as a renunciation of the doctrine of pre-determination; a thoughtful analysis shows that another and more subtle idea is involved. The central idea is that physical quantities are *related*, but are not all computable from a number of *independent basic* quantities (such as position and velocity).[6]

We shall show in §12 that this situation has an exact algebraic analogue in the calculus of propositions.

5. **Propositional calculi in classical dynamics.** Thus we see that an uncritical acceptance of the ideas of classical dynamics (particularly as they involve n-body problems) leads one to identify each subset of phase-space with an experimental proposition (the proposition that the system considered has position and momentum coördinates satisfying certain conditions) and conversely.

This is easily seen to be unrealistic; for example, how absurd it would be to call an "experimental proposition," the assertion that the angular momentum (in radians per second) of the earth around the sun was at a particular instant a rational number!

Actually, at least in statistics, it seems best to assume that it is the *Lebesgue-measurable* subsets of a phase-space which correspond to experimental propositions, two subsets being identified, if their difference has *Lebesgue-measure* 0.[7]

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[5] Like systems idealizing the solar system or projectile motion.

[6] A similar situation arises when one tries to correlate polarizations in different planes of electromagnetic waves.

[7] Cf. J. von Neumann, *"Operatorenmethoden in der klassischen Mechanik,"* Annals of Math. 33 (1932), 595–8. The difference of two sets S_1, S_2 is the set $(S_1 + S_2) - S_1 \cdot S_2$ of those points, which belong to one of them, but not to both.

But in either case, the set-theoretical sum and product of any two subsets, and the complement of any one subset of phase-space corresponding to experimental propositions, has the same property. That is, by definition[8]

The experimental propositions concerning any system in classical mechanics, correspond to a "field" of subsets of its phase-space. More precisely: To the "quotient" of such a field by an ideal in it. At any rate they form a "Boolean Algebra."[9]

In the axiomatic discussion of propositional calculi which follows, it will be shown that this is inevitable when one is dealing with exclusively compatible measurements, and also that it is logically immaterial which particular field of sets is used.

6. **A propositional calculus for quantum mechanics.** The question of the connection in quantum mechanics between subsets of observation-spaces (or "experimental propositions") and subsets of the phase-space of a system $\mathfrak{S}$, has not been touched. The present section will be devoted to defining such a connection, proving some facts about it, and obtaining from it heuristically by introducing a plausible postulate, a propositional calculus for quantum mechanics.

Accordingly, let us observe that if $\alpha_1, \cdots, \alpha_n$ are any compatible observations on a quantum-mechanical system $\mathfrak{S}$ with phase-space Σ, then[10] there exists a set of mutually orthogonal closed linear subspaces Ω_i of Σ (which correspond to the families of proper functions satisfying $\alpha_1 f = \lambda_{i,1} f, \cdots, \alpha_n f = \lambda_{i,n} f$) such that *every* point (or function) $f \in \Sigma$ can be uniquely written in the form

$$f = c_1 f_1 + c_2 f_2 + c_3 f_3 + \cdots \quad [f_i \in \Omega_i]$$

Hence if we state the

DEFINITION: By the "mathematical representative" of a subset S of any observation-space (determined by compatible observations $\alpha_1, \cdots, \alpha_n$) for a quantum-mechanical system $\mathfrak{S}$, will be meant the set of all points f of the phase-space of $\mathfrak{S}$, which are linearly determined by proper functions f_k satisfying $\alpha_1 f_k = \lambda_1 f_k, \cdots, \alpha_n f_k = \lambda_n f_k$, where $(\lambda_1, \cdots, \lambda_n) \in S$.

Then it follows immediately: (1) that the mathematical representative of any experimental proposition is a closed linear subspace of Hilbert space (2) since all operators of quantum mechanics are Hermitian, that the mathematical representative of the *negative*[11] of any experimental proposition is the *orthogonal*

[8] F. Hausdorff, "*Mengenlehre*," Berlin, 1927, p. 78.

[9] M. H. Stone, "*Boolean Algebras and their application to topology*," Proc. Nat. Acad. 20 (1934), p. 197.

[10] Cf. von Neumann, op. cit., pp. 121, 90, or Dirac, op. cit., 17. We disregard complications due to the possibility of a continuous spectrum. They are inessential in the present case.

[11] By the "negative" of an experimental proposition (or subset S of an observation-space) is meant the experimental proposition corresponding to the set-complement of S in the same observation-space.

complement of the mathematical representative of the proposition itself (3) the following three conditions on two experimental propositions P and Q concerning a given type of physical system are equivalent:

(3a) The mathematical representative of P is a subset of the mathematical representative of Q.

(3b) P implies Q—that is, whenever one can predict P with certainty, one can predict Q with certainty.

(3c) For any statistical ensemble of systems, the probability of P is at most the probability of Q.

The equivalence of (3a)–(3c) leads one to regard the aggregate of the mathematical representatives of the experimental propositions concerning any physical system $\mathfrak{S}$, as representing mathematically the propositional calculus for $\mathfrak{S}$.

We now introduce the

POSTULATE: *The set-theoretical product of any two mathematical representatives of experimental propositions concerning a quantum-mechanical system, is itself the mathematical representative of an experimental proposition.*

REMARKS: This postulate would clearly be implied by the not unnatural conjecture that all Hermitian-symmetric operators in Hilbert space (phase-space) correspond to observables;[12] it would even be implied by the conjecture that those operators which correspond to observables coincide with the Hermitian-symmetric elements of a suitable operator-ring M.[13]

Now the closed linear sum $\Omega_1 + \Omega_2$ of any two closed linear subspaces Ω_i of Hilbert space, is the orthogonal complement of the set-product $\Omega_1' \cdot \Omega_2'$ of the orthogonal complements Ω_i' of the Ω_i; hence if one adds the above postulate to the usual postulates of quantum theory, then one can deduce that

The set-product and closed linear sum of any two, and the orthogonal complement of any one closed linear subspace of Hilbert space representing mathematically an experimental proposition concerning a quantum-mechanical system $\mathfrak{S}$, itself represents an experimental proposition concerning $\mathfrak{S}$.

This defines the calculus of experimental propositions concerning $\mathfrak{S}$, as a calculus with three operations and a relation of implication, which closely resembles the systems defined in §5. We shall now turn to the analysis and comparison of all three calculi from an axiomatic-algebraic standpoint.

II. ALGEBRAIC ANALYSIS

7. **Implication as partial ordering.** It was suggested above that in any physical theory involving a phase-space, the experimental propositions concern-

[12] I.e., that given such an operator α, one "could" find an observable for which the proper states were the proper functions of α.

[13] F. J. Murray and J. v. Neumann, *"On rings of operators,"* Annals of Math., 37 (1936), p. 120. It is shown on p. 141, loc. cit. (Definition 4.2.1 and Lemma 4.2.1), that the closed linear sets of a ring M—that is those, the "projection operators" of which belong to M—coincide with the closed linear sets which are invariant under a certain group of rotations of Hilbert space. And the latter property is obviously conserved when a set-theoretical intersection is formed.

ing a system $\mathfrak{S}$ correspond to a family of subsets of its phase-space Σ, in such a way that "x implies y" (x and y being any two experimental propositions) means that the subset of Σ corresponding to x is contained set-theoretically in the subset corresponding to y. This hypothesis clearly is important in proportion as relationships of implication exist between experimental propositions corresponding to subsets of different observation-spaces.

The present section will be devoted to corroborating this hypothesis by identifying the algebraic-axiomatic properties of logical implication with those of set-inclusion.

It is customary to admit as relations of "implication," only relations satisfying

S1: x implies x.
S2: If x implies y and y implies z, then x implies z.
S3: If x implies y and y implies x, then x and y are logically equivalent.

In fact, S3 need not be stated as a postulate at all, but can be regarded as a definition of logical equivalence. Pursuing this line of thought, one can interpret as a "physical quality," the set of all experimental propositions logically equivalent to a given experimental proposition.[14]

Now if one regards the set S_x of propositions implying a given proposition x as a "mathematical representative" of x, then by $S3$ the correspondence between the x and the S_x is one-one, and x implies y if and only if $S_x \subset S_y$. While conversely, if L is any system of subsets X of a fixed class Γ, then there is an isomorphism which carries inclusion into logical implication between L and the system L^* of propositions "x is a point of X," $X \in L$.

Thus we see that the properties of logical implication are indistinguishable from those of set-inclusion, and that therefore it is *algebraically* reasonable to try to correlate physical qualities with subsets of phase-space.

A system satisfying S1-S3, and in which the relation "x implies y" is written $x \subset y$, is usually[15] called a "partially ordered system," and thus our first postulate concerning propositional calculi is that *the physical qualities attributable to any physical system form a partially ordered system.*

It does not seem excessive to require that in addition any such calculus contain two special propositions: the proposition $\square$ that the system considered *exists*, and the proposition $\circledcirc$ that it does *not exist*. Clearly

S4: $\circledcirc \subset x \subset \square$ for any x.

$\circledcirc$ is, from a logical standpoint, the "identically false" or "absurd" proposition; $\square$ is the "identically true" or "self-evident" proposition.

8. **Lattices.** In any calculus of propositions, it is natural to imagine that there is a weakest proposition implying, and a strongest proposition implied by,

[14] Thus in §6, closed linear subspaces of Hilbert space correspond one-many to experimental propositions, but one-one to physical qualities in this sense.

[15] F. Hausdorff, "*Grundzüge der Mengenlehre*," Leipzig, 1914, Chap. VI, §1.

a given pair of propositions. In fact, investigations of partially ordered systems from different angles all indicate that the first property which they are likely to possess, is the existence of greatest lower bounds and least upper bounds to subsets of their elements. Accordingly, we state

DEFINITION: A partially ordered system L will be called a "lattice" if and only if to any pair x and y of its elements there correspond

S5: A "meet" or "greatest lower bound" $x \cap y$ such that (5a) $x \cap y \subset x$, (5b) $x \cap y \subset y$, (5c) $z \subset x$ and $z \subset y$ imply $z \subset x \cap y$.

S6: A "join" or "least upper bound" $x \cap y$ satisfying (6a) $x \cup y \supset x$, (6b) $x \cup y \supset y$, (6c) $w \supset x$ and $w \supset y$ imply $w \supset x \cup y$.

The relation between meets and joins and abstract inclusion can be summarized as follows,[16]

(8.1) In any lattice L, the following formal identities are true,

L1: $a \cap a = a$ and $a \cup a = a$.
L2: $a \cap b = b \cap a$ and $a \cup b = b \cup a$.
L3: $a \cap (b \cap c) = (a \cap b) \cap c$ and $a \cup (b \cup c) = (a \cup b) \cup c$.
L4: $a \cup (a \cap b) = a \cap (a \cup b) = a$.

Moreover, the relations $a \supset b$, $a \cap b = b$, and $a \cup b = a$ are equivalent—each implies both of the others.

(8.2) Conversely, in any set of elements satisfying L2–L4 (L1 is redundant), $a \cap b = b$ and $a \cup b = a$ are equivalent. And if one defines them to mean $a \supset b$, then one reveals L as a lattice.

Clearly L1–L4 are well-known formal properties of *and* and *or* in ordinary logic. This gives an algebraic reason for admitting as a *postulate* (if necessary) the statement that a given calculus of propositions is a lattice. There are other reasons[17] which impel one to admit as a postulate the stronger statement that the set-product of any two subsets of a phase-space which correspond to physical qualities, itself represents a physical quality—this is, of course, the Postulate of §6.

It is worth remarking that in classical mechanics, one can easily define the meet or join of any two experimental propositions as an *experimental proposition*—simply by having independent observers read off the measurements which either proposition involves, and combining the results logically. This is true in quantum mechanics only exceptionally—only when all the measurements involved commute (are compatible); in general, one can only express the join or

[16] The final result was found independently by O. Öre, *"The foundations of abstract algebra. I.,"* Annals of Math. 36 (1935), 406–37, and by H. MacNeille in his Harvard Doctoral Thesis, 1935.

[17] The first reason is that this implies no restriction on the abstract nature of a lattice—any lattice can be realized as a system of its own subsets, in such a way that $a \cap b$ is the set-product of a and b. The second reason is that if one regards a subset S of the phase-space of a system $\mathfrak{S}$ as corresponding to the *certainty* of observing $\mathfrak{S}$ in S, then it is natural to assume that the combined certainty of observing $\mathfrak{S}$ in S and T is the certainty of observing $\mathfrak{S}$ in $S \cdot T = S \cap T$,—and assumes quantum theory.

meet of two given experimental propositions as a class of logically equivalent experimental propositions—i.e., as a *physical quality*.[18]

9. **Complemented lattices.** Besides the (binary) operations of meet- and join-formation, there is a third (unary) operation which may be defined in partially ordered systems. This is the operation of *complementation*.

In the case of lattices isomorphic with "fields" of sets, complementation corresponds to passage to the set-complement. In the case of closed linear subspaces of Hilbert space (or of Cartesian n-space), it corresponds to passage to the orthogonal complement. In either case, denoting the "complement" of an element a by a', one has the formal identities,

L71: $(a')' = a$.
L72: $a \cap a' = \circledcirc$ and $a \cup a' = \square$.
L73: $a \subset b$ implies $a' \supset b'$.

By definition, L71 and L73 amount to asserting that complementation is a "dual automorphism" of period two. It is an immediate corollary of this and the duality between the definitions (in terms of inclusion) of meet and join, that

L74: $(a \cap b)' = a' \cup b'$ and $(a \cup b)' = a' \cap b'$

and another corollary that the second half of L72 is redundant. [*Proof*: by L71 and the first half of L74, $(a \cup a') = (a'' \cup a') = (a' \cap a)' = \circledcirc'$, while under inversion of inclusion $\circledcirc$ evidently becomes $\square$.] This permits one to deduce L72 from the even weaker assumption that $a \subset a'$ implies $a = \circledcirc$. *Proof*: for any x, $(x \cap x')' = (x' \cup x'') = x' \cup x \supset x \cap x'$.

Hence if one admits as a postulate the assertion that *passage from an experimental proposition a to its complement a' is a dual automorphism of period two, and a implies a' is absurd*, one has in effect admitted L71–L74.

This postulate is independently suggested (and L71 proved) by the fact the "complement" of the proposition that the readings $x_1, \cdots, x_n$ from a series of compatible observations $\mu_1, \cdots, \mu_n$ lie in a subset S of $(x_1, \cdots, x_n)$-space, is by definition the proposition that the readings lie in the set-complement of S.

10. **The distributive identity.** Up to now, we have only discussed formal features of logical structure which seem to be common to classical dynamics and the quantum theory. We now turn to the central difference between them—the *distributive identity* of the propositional calculus:

L6: $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ and $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$

which is a law in classical, but not in quantum mechanics.

[18] The following point should be mentioned in order to avoid misunderstanding: If a, b are two physical qualities, then $a \cup b$, $a \cap b$ and a' (cf. below) are physical qualities too (and so are $\circledcirc$ and $\square+$). But $a \subset b$ is not a physical quality; it is a relation between physical qualities.

From an axiomatic viewpoint, each half of L6 implies the other.[19] Further, either half of L6, taken with L72, implies L71 and L73, and to assume L6 and L72 amounts to assuming the usual definition of a Boolean algebra.[20]

From a deeper mathematical viewpoint, L6 is the characteristic property of set-combination. More precisely, every "field" of sets is isomorphic with a Boolean algebra, and conversely.[21] This throws new light on the well-known fact that the propositional calculi of classical mechanics are Boolean algebras.

It is interesting that L6 is also a logical consequence of the compatibility of the observables occurring in a, b, and c. That is, if observations are made by independent observers, and combined according to the usual rules of logic, one can *prove* L1–L4, L6, and L71–74.

These facts suggest that the distributive law *may* break down in quantum mechanics. That it *does* break down is shown by the fact that if a denotes the experimental observation of a wave-packet ψ on one side of a plane in ordinary space, a' correspondingly the observation of ψ on the other side, and b the observation of ψ in a state symmetric about the plane, then (as one can readily check):

$$b \cap (a \cup a') = b \cap \square = b > \circledcirc = (b \cap a) = (b \cap a') = (b \cap a) \cup (b \cap a')$$

REMARK: In connection with this, it is a salient fact that the *generalized* distributive law of logic:

$$\text{L6}^*\text{:} \quad \prod_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} \right) = \sum_{j(i)} \left(\prod_{i=1}^{m} a_{i,j(i)} \right)$$

breaks down in the quotient algebra of the field of Lebesgue measurable sets by the ideal of sets of Lebesgue measure 0, which is so fundamental in statistics and the formulation of the ergodic principle.[22]

11. **The modular identity.** Although closed linear subspaces of Hilbert space and Cartesian n-space need not satisfy L6 relative to set-products and closed linear sums, the formal properties of these operations are not confined to L1–L4 and L71–L73.

In particular, set-products and straight linear sums are known[23] to satisfy the so-called "modular identity."

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[19] R. Dedekind, *"Werke,"* Braunschweig, 1931, vol. 2, p. 110.

[20] G. Birkhoff, *"On the combination of subalgebras,"* Proc. Camb. Phil. Soc. 29 (1933), 441–64, §§23–4. Also, in any lattice satisfying L6, isomorphism with respect to inclusion implies isomorphism with respect to complementation; this need not be true if L6 is not assumed, as the lattice of linear subspaces through the origin of Cartesian n-space shows.

[21] M. H. Stone, *"Boolean algebras and their application to topology,"* Proc. Nat. Acad. 20 (1934), 197–202.

[22] A detailed explanation will be omitted, for brevity; one could refer to work of G. D. Birkhoff, J. von Neumann, and A. Tarski.

[23] G. Birkhoff, op. cit., §28. The proof is easy. One first notes that since $a \subset (a \cup b) \cap c$ if $a \subset c$, and $b \cap c \subset (a \cup b) \cap c$ in any case, $a \cup (b \cap c) \subset (a \cup b) \cap c$. Then one notes

L5: If $a \subset c$, then $a \cup (b \cap c) = (a \cup b) \cap c$.

Therefore (since the linear sum of any two finite-dimensional linear subspaces of Hilbert space is itself finite-dimensional and consequently closed) set-products and *closed* linear sums of the *finite dimensional* subspaces of any topological linear space such as Cartesian n-space or Hilbert space satisfy L5, too.

One can interpret L5 directly in various ways. First, it is evidently a restricted associative law on mixed joins and meets. It can equally well be regarded as a weakened distributive law, since if $a \subset c$, then $a \cup (b \cap c) = (a \cap c) \cup (b \cap c)$ and $(a \cup b) \cap c = (a \cup b) \cap (a \cup c)$. And it is self-dual: replacing $\subset, \cap, \cup$ by $\supset, \cup, \cap$ merely replaces a, b, c, by c, b, a.

Also, speaking graphically, the assumption that a lattice L is "modular" (i.e., satisfies L5) is equivalent to[24] saying that L contains no sublattice isomorphic with the lattice graphed in fig. 1:

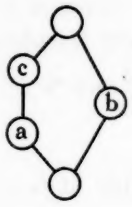

FIG. 1

Thus in Hilbert space, one can find a counterexample to L5 of this type. Denote by $\xi_1, \xi_2, \xi_3, \cdots$ a basis of orthonormal vectors of the space, and by a, b, and c respectively the closed linear subspaces generated by the vectors $(\xi_{2n} + 10^{-n}\xi_1 + 10^{-2n}\xi_{2n+1})$, by the vectors ξ_{2n}, and by a and the vector ξ_1. Then a, b, and c generate the lattice of Fig. 1.

Finally, the modular identity can be proved to be a consequence of the assumption that there exists a numerical dimension-function $d(a)$, with the properties

D1: If $a > b$, then $d(a) > d(b)$.
D2: $d(a) + d(b) = d(a \cap b) + d(a \cup b)$.

This theorem has a converse under the restriction to lattices in which there is a finite upper bound to the length n of chains[25] $\circledcirc < a_1 < a_2 < \cdots < a_n < \square$ of elements.

Since conditions D1–D2 partially describe the formal properties of probability, the presence of condition L5 is closely related to the existence of an

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that any vector in $(a \cup b) \cap c$ can be written $\xi = \alpha + \beta$ [$\alpha \in a, \beta \in b, \xi \in c$]. But $\beta = \xi - \alpha$ is in c (since $\xi \in c$ and $\alpha \in a \subset c$); hence $\xi = \alpha + \beta \in a \cup (b \cap c)$, and $a \cup (b \cap c) \supset (a \cup b) \cap c$, completing the proof.

[24] R. Dedekind, "*Werke*," vol. 2, p. 255.

[25] The statements of this paragraph are corollaries of Theorem 10.2 of G. Birkhoff, op. cit.

"a priori thermo-dynamic weight of states." But it would be desirable to interpret L5 by simpler phenomenological properties of quantum physics.

12. **Relation to abstract projective geometries.** We shall next investigate how the assumption of postulates asserting that the physical qualities attributable to any quantum-mechanical system $\mathfrak{S}$ are a lattice satisfying L5 and L71–L73 characterizes the resulting propositional calculus. This question is evidently purely algebraic.

We believe that the best way to find this out is to introduce an assumption limiting the length of chains of elements (assumption of finite dimensions) of the lattice, admitting frankly that the assumption is purely heuristic.

It is known[26] that any lattice of finite dimensions satisfying L5 and L72 is the direct product of a finite number of abstract projective geometries (in the sense of Veblen and Young), and a finite Boolean algebra, and conversely.

REMARK: It is a corollary that a lattice satisfying L5 and L71–L73 possesses independent basic elements of which any element is a union, if and only if it is a Boolean algebra.

Again, such a lattice is a single projective geometry if and only if it is *irreducible*—that is, if and only if it contains no "neutral" elements.[27] $x \neq \circledcirc, \square$ such that $a = (a \cap x) \cup (a \cap x')$ for all a. In actual quantum mechanics such an element would have a projection-operator, which commutes with all projection-operators of observables, and so with all operators of observables in general. This would violate the requirement of "irreducibility" in quantum mechanics.[28] Hence we conclude that the *propositional calculus of quantum mechanics has the same structure as an abstract projective geometry.*

Moreover, this conclusion has been obtained purely by analyzing internal properties of the calculus, in a way which involves Hilbert space only indirectly.

13. **Abstract projective geometries and skew-fields.** We shall now try to get a fresh picture of the propositional calculus of quantum mechanics, by recalling the well-known two-way correspondence between abstract projective geometries and (not necessarily commutative) fields.

Namely, let F be any such field, and consider the following definitions and constructions: n elements $x_1, \cdots, x_n$ of F, not all $= 0$, form a right-ratio $[x_1 : \cdots : x_n]_r$, two right-ratios $[x_1 : \cdots : x_n]_r$, and $[\xi_1 : \cdots : \xi_n]_r$ being called "equal," if and only if a $z \in F$ with $\xi_i = x_i z$, $i = 1, \cdots, n$, exists. Similarly, n elements $y_1, \cdots, y_n$ of F, not all $= 0$, form a left-ratio $[y_1 : \cdots : y_n]_l$, two left-ratios $[y_1 : \cdots : y_n]_l$ and $[\eta_1 : \cdots : \eta_n]_l$ being called "equal," if and only if a z in F with $\eta_i = z y_i$, $i = 1, \cdots, n$, exists.

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[26] G. Birkhoff *"Combinatorial relations in projective geometries,"* Annals of Math. 36 (1935), 743–8.

[27] O. Öre, op. cit., p. 419.

[28] Using the terminology of footnote,[13] and of loc. cit. there: The ring MM' should contain no other projection-operators than 0, 1, or: the ring M must be a "factor." Cf. loc. cit.[13], p. 120.

Now define an $n-1$-dimensional projective geometry $P_{n-1}(F)$ as follows: The "points" of $P_{n-1}(F)$ are all right-ratios $[x_1:\cdots:x_n]_r$. The "linear subspaces" of $P_{m-1}(F)$ are those sets of points, which are defined by systems of equations

$$\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n = 0, \qquad k = 1, \cdots, m.$$

($m = 1, 2, \cdots$, the α_{ki} are fixed, but arbitrary elements of F). The proof, that this *is* an abstract projective geometry, amounts simply to restating the basic properties of linear dependence.[29]

The same considerations show, that the ($n-2$-dimensional) hyperplanes in $P_{m-1}(F)$ correspond to $m = 1$, not all $\alpha_i = 0$. Put $\alpha_{1i} = y_i$, then we have

$$(*) \qquad y_1x_1 + \cdots + y_nx_n = 0, \qquad \text{not all } y_i = 0.$$

This proves, that the ($n-2$-dimensional) hyperplanes in $P_{m-1}(F)$ are in a one-to-one correspondence with the left-ratios $[y_r:\cdots:y_n]_l$.

So we can identify them with the left-ratios, as points are already identical with the right-ratios, and (*) becomes the definition of "incidence" (point $\subset$ hyperplane).

Reciprocally, any abstract $n-1$-dimensional projective geometry Q_{n-1} with $n = 4, 5, \cdots$ belongs in this way to some (not necessarily commutative field $F(Q_{n-1})$, and Q_{n-1} is isomorphic with $P_{n-1}(F(Q_{n-1}))$.[30]

14. **Relation of abstract complementarity to involutory anti-isomorphisms in skew-fields.** We have seen that the family of irreducible lattices satisfying L5 and L72 is precisely the family of projective geometries, provided we exclude the two-dimensional case. But what about L71 and L73? In other words, for which $P_{n-1}(F)$ can one define complements possessing all the known formal properties of orthogonal complements? The present section will be spent in answering this question.[30a]

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[29] Cf. §§103–105 of B. L. Van der Waerden's "*Moderne Algebra*," Berlin, 1931, Vol. 2.

[30] $n = 4, 5, \cdots$ means of course $n - 1 \geqq 3$, that is, that Q_{n-1} is necessarily a "Desarguesian" geometry. (Cf. O. Veblen and J. W. Young, "*Projective Geometry*," New York, 1910, Vol. 1, page 41). Then $F = F(Q_{n-1})$ can be constructed in the classical way. (Cf. Veblen and Young, Vol. 1, pages 141–150). The proof of the isomorphism between Q_{n-1} and the $P_{n-1}(F)$ as constructed above, amounts to this: Introducing (not necessarily commutative) homogeneous coördinates $x_1, \cdots, x_n$ from F in Q_{n-1}, and expressing the equations of hyperplanes with their help. This can be done in the manner which is familiar in projective geometry, although most books consider the commutative ("Pascalian") case only. D. Hilbert, "*Grundlagen der Geometrie*," 7th edition, 1930, pages 96–103, considers the non-commutative case, but for affine geometry, and $n - 1 = 2, 3$ only.

Considering the lengthy although elementary character of the complete proof, we propose to publish it elsewhere.

[30a] R. Brauer, "*A characterization of null systems in projective space*," Bull. Am. Math. Soc. 42 (1936), 247–54, treats the analogous question in the opposite case that $X \cap X' \neq 0$ is postulated.

First, we shall show that it is *sufficient* that F admit an *involutory antisomorphism* $W: \bar{x} = W(x)$, that is:

Q1. $w(w(u)) = u$,
Q2. $w(u + v) = w(u) + w(v)$,
Q3. $w(uv) = w(v)\, w(u)$,

with a *definite diagonal Hermitian form* $w(x_1)\gamma_1\xi_1 + \cdots + w(x_n)\gamma_n\xi_n$, where

Q4. $w(x_1)\gamma_1 x_1 + \cdots + w(x_n)\gamma_n x_n = 0$ implies $x_1 = \cdots = x_n = 0$,

the γ_i being fixed elements of F, satisfying $w(\gamma_i) = \gamma_i$.

Proof: Consider ennuples (not right- or left-ratios!) $x: (x_1, \cdots, x_n)$, $\xi: (\xi_1, \cdots, \xi_n)$ of elements of F. Define for them the vector-operations

$$xz: (x_1 z, \cdots, x_n z) \qquad (z \text{ in } F),$$

$$x + \xi: (x_1 + \xi_1, \cdots, x_n + \xi_n),$$

and an "inner product"

$$(\xi_1 x) = w(\xi_1)\gamma_1 x_1 + \cdots + w(\xi_n)\gamma_n x_n.$$

Then the following formulas are corollaries of Q1–Q4.

IP1 $(x, \xi) = w((\xi, x))$,
IP2 $(\xi, xu) = (\xi, x)u$, $(\xi u, x) = w(u)(\xi, x)$,
IP3 $(\xi, x' + x'') = (\xi, x') + (\xi, x'')$, $(\xi' + \xi'', x) = (\xi', x) + (\xi'', x)$,
IP4 $(x, x) = w((x, x)) = [x]$ is $\neq 0$ if $x \neq 0$ (that is, if any $x_i \neq 0$).

We can define $x \perp \xi$ (in words: "x is orthogonal to ξ") to mean that $(\xi, x) = 0$. This is evidently symmetric in x, ξ, and depends on the right-ratios $[x_1 : \cdots : x_n]_r$, $[\xi_1 : \cdots : \xi_n]_r$ only so it establishes the relation of "polarity," $a \perp b$, between the points

$$a: [x_1 : \cdots : x_n]_r, \qquad b: [\xi_1 : \cdots : \xi_n]_r \text{ of } P_{n-1}(F).$$

The polars to any point b: $[\xi_1 : \cdots : \xi_n]_r$ of $P_{n-1}(F)$ constitute a linear subspace of points of $P_{n-1}(F)$, which by Q4 does not contain b itself, and yet with b generates whole projective space $P_{n-1}(F)$, since for any ennuple $x: (x_1, \cdots, x_n)$

$$x = x' + \xi \cdot [\xi]^{-1}(\xi, x)$$

where by Q4, $[\xi] \neq 0$, and by IP $(\xi, x') = 0$. This linear subspace is, therefore, an n-2-dimensional hyperplane.

Hence if c is any k-dimensional element of $P_{n-1}(F)_1$ one can set up inductively k mutually polar points $b^{(1)}, \cdots, b^{(k)}$ in c. Then it is easy to show that the set c' of points polar to every $b^{(1)}, \cdots, b^{(k)}$—or equivalently to every point in c—constitute an n-k-1-dimensional element, satisfying $c \cap c' = \circledcirc$ and $c \cup c' = \square$. Moreover, by symmetry $(c')' \supset c$, whence by dimensional considerations $c'' = c$. Finally, $c \supset d$ implies $c' \subset d'$, and so the correspondence $c \to c'$ defines an involutory dual automorphism of $P_{n-1}(F)$ completing the proof.

In the Appendix it will be shown that this condition is also necessary. Thus the above class of systems is exactly the class of irreducible lattices of finite dimensions > 3 satisfying L5 and L71–L73.

III. Conclusions

15. **Mathematical models for propositional calculi.** One conclusion which can be drawn from the preceding algebraic considerations, is that one can construct many different models for a propositional calculus in quantum mechanics, which cannot be differentiated by known criteria. More precisely, one can take any field F having an involutory anti-isomorphism satisfying Q4 (such fields include the real, complex, and quaternion number systems[31]), introduce suitable notions of linear dependence and complementarity, and then construct for every dimension-number n a model $P_n(F)$, having all of the properties of the propositional calculus suggested by quantum-mechanics.

One can also construct infinite-dimensional models $P_\infty(F)$ whose elements consist of all closed linear subspaces of normed infinite-dimensional spaces. But philosophically, Hankel's principle of the "perseverance of formal laws" (which leads one to try to preserve L5)[32] and mathematically, technical analysis of spectral theory in Hilbert space, lead one to prefer a continuous-dimensional model $P_c(F)$, which will be described by one of us in another paper.[33]

$P_c(F)$ is very analogous with the model furnished by the measurable subsets of phase-space in classical dynamics.[34]

16. **The logical coherence of quantum mechanics.** The above heuristic considerations suggest in particular that the physically significant statements in quantum mechanics actually constitute a sort of projective geometry, while the physically significant statements concerning a given system in classical dynamics constitute a Boolean algebra.

They suggest even more strongly that whereas in classical mechanics any propositional calculus involving more than two propositions can be decomposed into independent constituents (direct sums in the sense of modern algebra), quantum theory involves irreducible propositional calculi of unbounded complexity. This indicates that quantum mechanics has a *greater logical coherence*

[31] In the real case, $w(x) = x$; in the complex case, $w(x + iy) = x - iy$; in the quaternionic case, $w(u + ix + jy + kz) = u - ix - jy - kz$; in all cases, the λ_i are 1. Conversely, A. Kolmogoroff, "*Zur Begründung der projektiven Geometrie*," Annals of Math. 33 (1932), 175–6 has shown that any projective geometry whose k-dimensional elements have a *locally compact topology* relative to which the lattice operations are continuous, must be over the real, the complex, or the quaternion field.

[32] L5 can also be preserved by the artifice of considering in $P_\infty(F)$ only elements which either are or have complements which are of finite dimensions.

[33] J. von Neumann, "*Continuous geometries*," Proc. Nat. Acad., 22 (1936), 92–100 and 101–109. These may be a more suitable frame for quantum theory, than Hilbert space.

[34] In quantum mechanics, dimensions but not complements are uniquely determined by the inclusion relation; in classical mechanics, the reverse is true!

than classical mechanics—a conclusion corroborated by the impossibility in general of measuring different quantities independently.

17. **Relation to pure logic.** The models for propositional calculi which have been considered in the preceding sections are also interesting from the standpoint of pure logic. Their nature is determined by quasi-physical and technical reasoning, different from the introspective and philosophical considerations which have had to guide logicians hitherto. Hence it is interesting to compare the modifications which they introduce into Boolean algebra, with those which logicians on "intuitionist" and related grounds have tried introducing.

The main difference seems to be that whereas logicians have usually assumed that properties L71–L73 of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the *distributive identities* L6 as the weakest link in the algebra of logic. Cf. the last two paragraphs of §10.

Our conclusion agrees perhaps more with those critiques of logic, which find most objectionable the assumption that $a' \cup b = \square$ implies $a \subset b$ (or dually, the assumption that $a \cap b' = \circledcirc$ implies $b \supset a$—the assumption that to deduce an absurdity from the conjunction of a and not b, justifies one in inferring that a implies b).[35]

18. **Suggested questions.** The same heuristic reasoning suggests the following as fruitful questions.

What experimental meaning can one attach to the meet and join of two given experimental propositions?

What simple and plausible physical motivation is there for condition L5?

Appendix

1. Consider a projective geometry Q_{n-1} as described in §13. F is a (not necessarily commutative, but associative) field, $n = 4, 5, \cdots$, $Q_{n-1} = P_{n-1}(F)$ the projective geometry of all right-ratios $[x_1 : \cdots : x_n]_r$, which are the *points* of Q_{n-1}. The ($n - 2$-dimensional) *hyperplanes* are represented by the left-ratios $[y_1 : \cdots y_n]_l$, incidence of a point $[x_1 : \cdots x_n]_r$ and of a hyperplane $[y_1 : \cdots : y_n]_l$ being defined by

$$\sum_{i=1}^{n} y_i x_i = 0 \tag{1}$$

All linear subspaces of Q_{n-1} form the lattice L, with the elements $a, b, c, \cdots$.

Assume now that an operation a' with the properties L71–L73 in §9 exists:

L71 $(a')' = a$

L72 $a \cap a' = \circledcirc$ and $a \cup a' = \square$,

L73 $a \subset b$ implies $a' \supset b'$.

[35] It is not difficult to show, that assuming our axioms L1–5 and 7, the distributive law L6 is equivalent to this postulate: $a' \cup b = \square$ implies $a \subset b$.

They imply (cf. §9)

$$\text{L74} \quad (a \cap b)' = a' \cup b' \text{ and } (a \cup b)' = a' \cap b'.$$

Observe, that the relation $a \subset b'$ is symmetric in a, b, owing to L73 and L71.

2. If a: $[x_1 : \cdots : x_n]_r$ is a point, then a' is an $[y_1 : \cdots y_n]_l$. So we may write:

$$[x_1 : \cdots : x_n]_r' = [y_1 : \cdots : \eta_n]_l, \tag{2}$$

and define an operation which connects right- and left-ratios. We know from §14, that a general characterization of a' (a any element of L) is obtained, as soon as we derive an algebraic characterization of the above $[x_1 : \cdots : x_n]_r'$. We will now find such a characterization of $[x_1 : \cdots : x_n]_r'$, and show, that it justifies the description given in §14.

In order to do this, we will have to make a rather free use of *collineations* in Q_{n-1}. A collineation is, by definition, a coördinate-transformation, which replaces $[x_1 : \cdots : x_n]_r$ by $[\bar{x}_1 : \cdots : \bar{x}_n]_r$,

$$\bar{x}_j = \sum_{i=2}^{n} \omega_{ij}\, x_i \qquad \text{for } j = 1, \cdots, n. \tag{3}$$

Here the ω_{ij} are fixed elements of F, and such, that (3) has an inverse.

$$x_i = \sum_{j=1}^{n} \theta_{ij}\, \bar{x}_j, \qquad \text{for } i = 1, \cdots, n, \tag{4}$$

the θ_{ij} being fixed elements of F, too. (3), (4) clearly mean

$$\delta_{kl} = \left\{ \begin{matrix} 1 \text{ if } k = 1 \\ 0 \text{ if } k \neq 1 \end{matrix} \right\}:$$

$$\sum_{j=1}^{n} \theta_{ij}\, \omega_{kj} = \delta_{ik}, \qquad \sum_{i=1}^{n} \omega_{ij}\, \theta_{ik} = \delta_{jk}. \tag{5}$$

Considering (1) and (5) they imply the contravariant coördinate-transformation for hyperplanes: $[y_1 : \cdots : y_n]_l$ becomes $[\bar{y}_1 : \cdots : \bar{y}_n]_l$, where

$$\bar{y}_i = \sum_{i=1}^{n} y_i\, \theta_{ij}, \qquad \text{for } j = 1, \cdots, n, \tag{6}$$

$$y_i = \sum_{j=1}^{n} \bar{y}_j\, \omega_{ij}, \qquad \text{for } i = 1, \cdots, n. \tag{7}$$

(Observe, that the position of the coefficients on the left side of the variables in (4), (5), and on their right side in (6), (7), is essential!)

3. We will bring about

$$[\delta_{i1} : \cdots : \delta_{in}]_r' = [\delta_{i1} : \cdots : \delta_{in}]_l \qquad \text{for } i = 1, \cdots, n, \tag{8}$$

by choosing a suitable system of coördinates, that is, by applying suitable collineations. We proceed by induction: Assume that (8) holds for $i = 1, \cdots, m-1 (m = 1, \cdots, n)$, then we shall find a collineation which makes (8) true for $i = 1, \cdots, m$.

Denote the point $[\delta_{i1}: \cdots : \delta_{in}]_r$ by p_i^*, and the hyperplane $[\delta_{i1}: \cdots : \delta_{in}]_l'$ by h_i^* our assumption on (8) is: $p_i^{*\prime} = h_i^*$ for $i = 1, \cdots, m-1$. Consider now a point a: $[x_1: \cdots : x_n]_r$, and the hyperplane a': $[y_1: \cdots : y_n]_l$. Now $a \leqq p_i^{*\prime} = h_i^*$ means (use (1)) $x_i = 0$, and $p_i^* \leqq a'$ means (use (8)) $y_i = 0$. But these two statements are equivalent. So we see: If $i = 1, \cdots, m-1$, then $x_i = 0$ and $y_i = 0$ are equivalent.

Consider now p_m^*: $[\delta_{m1}: \cdots : \delta_{mn}]_r$. Put p_m': $[y_1^*: \cdots : y_n^*]_l$. As $\delta_{mi} = 0$ for $i = 1, \cdots, m-1$, so we have $y_i^* = 0$ for $i = 1, \cdots, m-1$. Furthermore, $p_m^* \cap p_m^{*\prime} = 0$, $p_m^* \neq 0$, so p_m^* not $\leqq p_m^{*\prime}$. By (1) this means $y_m^* \neq 0$.

Form the collineation (3), (4), (6), (7), with

$$\theta_{ii} = \omega_{ii} = 1, \qquad \theta_{mi} = \omega_{im} = y_m^{*-1} y_i^* \quad \text{for } i = m+1, \cdots, n,$$

all other $\theta_{ij}, \omega_{ij} = 0$.

One verifies immediately, that this collineation leaves the coördinates of the p_1^*: $[\delta_{i1}: \cdots : \delta_{in}]_r$, $i = 1, \cdots, n$, invariant, and similarly those of the $p_i^{*\prime}$: $[\delta_{i1}: \cdots : \delta_{im}]_l$, $i = 1, \cdots m-1$, while it transforms those of

$$p_m^{*\prime}: [y_1^*: \cdots : y_n^*]_l$$

into $[\delta_{m1}: \cdots : \delta_{mn}]_l$.

So after this collineation (8) holds for $i = 1, \cdots, m$.

Thus we may assume, by induction over $m = 1, \cdots, n$, that (8) holds for all $i = 1, \cdots, n$. This we will do.

The above argument now shows, that for a: $[x_1: \cdots : x_n]_r$, a': $[y_1: \cdots : y_n]_l$,

(9) $\qquad x_i = 0$ is equivalent to $y_i = 0$, $\qquad$ for $i = 1, \cdots, n$.

4. Put a: $[x_1: \cdots : x_n]_r$, a': $[y_1: \cdots : y_n]_l$, and b: $[\xi_1: \cdots : \xi_n]_r$, b': $[\eta_1: \cdots : \eta_n]_l$.

Assume first $\eta_1 = 1$, $\eta_2 = \eta$, $\eta_3 = \cdots = \eta_n = 0$. Then (9) gives $\xi_1 \neq 0$, so we can normalize $\xi_1 = 1$, and $\xi_3 = \cdots = \xi_n = 0$. ξ_2 can depend on $\eta_2 = \eta$ only, so $\xi_2 = f_2(\eta)$.

Assume further $x_1 = 1$. Then (9) gives $y_1 \neq 0$, so we can normalize $y_1 = 1$. Now $a \leqq b'$ means by (i) $1 + \eta x_2 = 0$, and $b \leqq a'$ means $1 + y_2 f_2(\eta) = 0$. These two statements must, therefore, be equivalent. So if $x_2 \neq 0$, we may put $\eta = -x_2^{-1}$, and obtain $y_2 = -(f_2(\eta))^{-1} = -(f_2(-x_2^{-1}))^{-1}$. If $x_2 = 0$, then $y_2 = 0$ by (9). Thus, x_2 determines at any rate y_2 (independently of $x_3, \cdots, x_n$): $y_2 = \varphi_2(x_2)$. Permuting the $i = 2, \cdots, n$ gives, therefore:

There exists for each $i = 2, \cdots, n$ a function $\varphi_i(x)$, such that $y_i = \varphi_i(x_i)$. Or:

(10) $\qquad$ If a: $[1: x_2: \cdots : x_n]_r$, $\qquad$ then a': $[1: \varphi_2(x_2): \cdots : \varphi_n(x_n)]_l$.

Applying this to a: $[1:x_2:\cdots:x_n]_r$ and c: $[1:u_1:\cdots:u_n]_r$ shows: As $a \leqq c'$ and $c \leqq a'$ are equivalent, so

$$\sum_{i=2}^{n} \varphi_i(u_i)\, x_i = -1 \text{ is equivalent to } \sum_{i=2}^{n} \varphi(x_i)\, u_i = -1. \tag{11}$$

Observe, that (9) becomes:

$$\varphi_i(x) = 0 \text{ if and only if } x = 0. \tag{12}$$

5. (11) with $x_3 = \cdots = x_n = u_3 = \cdots = u_n = 0$ shows: $\varphi_2(u_2)x_2 = -1$ is equivalent to $\varphi_2(x_2)u_2 = -1$. If $x_2 \neq 0$, $u_2 = (-\varphi_2(x_2))^{-1}$, then the second equation holds, and so both do.

Choose x_2, u_2 in this way, but leave $x_3, \cdots, x_n, u_3, \cdots u_n$ arbitrary. Then (11) becomes:

$$\sum_{i=3}^{n} \varphi_i(u_i)\, x_i = 0 \text{ is equivalent to } \sum_{i=3}^{n} \varphi_i\,(x_i)\, u_i = 0. \tag{13}$$

Now put $x_5 = \cdots = x_n = u_5 = \cdots = u_n = 0$. Then (13) becomes:

$$\varphi_3(u_3)x_3 + \varphi_4(u_4)x_4 = 0 \text{ is equivalent to } \varphi_3(x_3)u_3 + \varphi_4(x_4)u_4 = 0,$$

that is (for $x_4, u_4 \neq 0$):

$$\text{(a)} \quad x_3x_4^{-1} = \varphi_4(u_4)^{-1}\,\varphi_3(u_3)$$

(14) is equivalent to

$$\text{(b)} \quad u_3u_4^{-1} = \varphi_4(x_4)^{-1}\,\varphi_3(x_3).$$

Let x_4, x_3 be given. Choose u_3, u_4 so as to satisfy (b). Then (a) is true, too. Now (a) remains true, if we leave u_3, u_4 unchanged, but change x_3, x_4 without changing $x_3x_4^{-1}$. So (b) remains too true under these conditions, that is, the value of $\varphi_4(x_4)^{-1}\,\varphi_3(x_3)$ does not change. In other words: $\varphi_4(x_4)^{-1}\,\varphi_3(x_3)$ depends on $x_3x_4^{-1}$ only. That is: $\varphi_4(x_4)^{-1}\,\varphi_3(x_3) = \varphi_{34}(x_3x_4^{-1})$. Put $x_3 = xz$, $x_4 = x$, then we obtain:

$$\varphi_3(xz) = \varphi_4(x)\,\psi_{34}(z). \tag{15}$$

This was derived for $x, z \neq 0$, but it will hold for x or $z = 0$, too, if we define $\psi_{34}(0) = 0$. (Use (12).)

(15), with $z = 1$ gives $\varphi_3(x) = \varphi_4(x)\alpha_{34}$, where $\alpha_{34} = \psi_{34}(1) \neq 0$, owing to (12) for $x \neq 0$. Permuting the $i = 2, \cdots, n$ gives, therefore:

$$\varphi_i(x) = \varphi_j(x)\alpha_{ij}, \qquad \text{where } \alpha_{ij} \neq 0. \tag{16}$$

(For $i = j$ put $\alpha_{ii} = 1$.)

Now (15) becomes

$$\begin{aligned} \varphi_2(zx) &= \varphi_2(x)w(z) \\ w(z) &= \alpha_{42}\psi_{34}(z)\alpha_{23}. \end{aligned} \tag{17}$$

Put $x = 1$ in (17), write x for z, and use (16) with $j = 2$:

$$\varphi_i(x) = \beta w(z)\gamma_i, \qquad \text{where } \beta, \gamma_i \neq 0. \tag{18}$$
$$(\beta = \varphi_2(1),\ \gamma_i = \alpha_{i2}).$$

6. Compare (17) for $x = 1, z = u$; $x = u, z = v$; and $x = 1, z = vu$. Then

$$w(vu) = w(u)w(v) \tag{19}$$

results (12) and (18) give

$$w(u) = 0 \text{ if and only if } u = 0. \tag{20}$$

Now write $w(z)$, γ_i for $\beta w(z)\beta^{-1}$, $\beta\gamma_i$. Then (18), (19), (20) remain true, (18) is simplified in so far, as we have $\beta = 1$ there. So (11) becomes

$$\sum_{i=2}^{n} w(u_i)\,\gamma_i\, x = -1 \tag{21}$$

(21) is equivalent to

$$\sum_{i=2}^{n} w(x_i)\,\gamma_i\, u_i = -1$$

$x_2 = x$, $u_2 = u$ and all other $x_i = u_i = 0$ give: $w(u)\gamma_2 x = -1$ is equivalent to $w(x)\gamma_2 u = -1$. If $x \neq 0$, $u = -\gamma_2^{-1}\, w(x)^{-1}$, then the second equation holds, and so the first one gives: $x = -\gamma_2^{-1}\, w(u)^{-1} = -\gamma_2^{-1}(w(-\gamma_2^{-1}w(x)^{-1}))^{-1}$. But (19), (20) imply $w(1) = 1$, $w(w^{-1}) = w(w)^{-1}$, so the above relation becomes:

$$x = -\gamma_2^{-1}(w(-\gamma_2^{-1}w(x_1^{-1}))^{-1} = -\gamma_2^{-1}w((-\gamma_2^{-1}w(x)^{-1})^{-1})$$
$$= -\gamma_2^{-1}w(w(x)(-\gamma_2)) = -\gamma_2^{-1}w(-\gamma_2)w(w(x)).$$

Put herein $x = 1$, as $w(w(1)) = w(1) = 1$, so $-\gamma_2^{-1}w(-\gamma_2) = 1$, $w(-\gamma_2) = -\gamma_2$ results. Thus the above equation becomes

$$w(w(x)) = x, \tag{22}$$

and $w(-\gamma_2) = -\gamma_2$ gives, if we permute the $i = 2, \cdots, n$,

$$w(-\gamma_i) = -\gamma_i. \tag{23}$$

Put $u_i = -\gamma_i^{-1}$ in (21). Then considering (22) and (19)

$$\sum_{i=2}^{n} x_i = 1 \text{ is equivalent to } \sum_{i=2}^{n} w(x_i) = 1 \tag{24}$$

obtains. Put $x_2 = x$, $x_3 = y$, $x_4 = 1 - x - y$, $x_5 = \cdots = x_n = 0$. Then (24) gives $w(x) + w(y) = 1 - w(1 - x - y)$. So $w(x) + w(y)$ depends on $x + y$ only. Replacing x, y by $x + y$, 0 shows, that it is equal to $w(x + y) + w(0) = w(x + y)$ (use 20). So we have:

$$w(x) + w(y) = w(x + y) \tag{25}$$

(25), (19) and (22) give together:

$w(x)$ is an involutory antisomorphism of F.

Observe, that (25) implies $w(-1) = -w(1) = -1$, and so (23) becomes

$$w(\gamma_i) = \gamma_i. \tag{26}$$

7. Consider a: $[x_1:\cdots:x_n]_r$, a': $[y_1:\cdots:y_n]_l$. If $x_1 \neq 0$, we may write a: $[1:x_2x_1^{-1}:\cdots:x_nx_1^{-1}]_r$, and so a': $[1:w(x_2x_1^{-1})\gamma_2:\cdots:w(x_nx_1^{-1})\gamma_n]_l$. But

$$w(x_ix_1^{-1})\gamma_i = w(x_1^{-1})w(x_i)\gamma_i = w(x_1)^{-1}w(x_i)\gamma_i,$$

and so we can write

$$a':[w(x_1):w(x_2)\gamma_2:\cdots:w(x_n)\gamma_n]_l$$

too. So we have

$$y_i = w(x_i)\gamma_i \qquad \text{for } i = 1, \cdots, n, \tag{27}$$

where the γ_i for $i = 2, \cdots, n$ are those from **6.**, and $\gamma_1 = 1$. And $w(1) = 1$, so (26) holds for all $i = 1, \cdots, n$. So we have the representation (27) with γ_i obeying (26), if $x_i \neq 0$.

Permutation of the $i = 1, \cdots, n$ shows, that a similar relation holds if $x_2 \neq 0$:

$$y_i = w^+(x_i)\gamma_i^+, \tag{27$^+$}$$

$$w^+(\gamma_i^+) = \gamma_i^+, \tag{26$^+$}$$

$w^+(x)$ being an involutory antisomorphism of F. ($w^+(x)$, γ_i^+ may differ from $w(x)$, γ_i!) Instead of $\gamma_1 = 1$ we have now $\gamma_2^+ = 1$, but we will not use this.

Put all $x_i = 1$. Then a': $[y_1:\cdots:y_n]_l$ can be expressed by both formulae (27) and (27$^+$). As $w(x)_1w^+(x)$ are both antisomorphism, so $w(1) = w^+(1) = 1$, and therefore $[y_1:\cdots:y_n]_l = [\gamma_1:\cdots:\gamma_n]_l = [\gamma_1^+:\cdots:\gamma_n^+]_l$ obtains. Thus $(\gamma_1^+)^{-1}\gamma_i^+ = (\gamma_1)^{-1}\gamma_i = \gamma_i$, $\gamma_i^+ = \gamma_1^+\gamma_i$ for $i = 1, \cdots, n$.

Assume now $x_2 \neq 0$ only. Then (27$^+$) gives $y_i = w^+(x_i)\gamma_i^+$, but as we are dealing with left ratios, we may as well put

$$y_i = (\gamma_1^+)^{-1}w^+(x_i)\gamma_i^+ = (\gamma_1{}^+)^{-1}w^+(x)\gamma_1^+\gamma_i.$$

Put $\beta^+ = \gamma_1^+ \neq 0$, then we have:

$$y_i = \beta^{+-1}w^+(x_i)\beta^+\gamma_i. \tag{27^{++}}$$

Put now $x_1 = x_2 = 1$, $x_3 = x$, all other $x_i = 0$. Again a': $[y_1:\cdots:y_n]_l$ can be expressed by both formulae (27) and (27^{++}), again $w(1) = w^+(1)$. Therefore

$$[y_1:y_2:y_3:y_4:\cdots:y_n]_l = [\gamma_1:\gamma_2:w(x)\gamma_3:0:\cdots:0]_l$$
$$= [\gamma_1:\gamma_2:\beta^{+-1}w^+(x)\beta^+\gamma_3:0:\cdots:0]_l$$

obtains. This implies $w(x) = \beta^{+-1}w(x)\beta^+$ for all x, and so (27^{++}) coincides with (27).

In other words: (27) holds for $x_2 \neq 0$ too.

Permuting $i = 2, \cdots, n$ (only $i = 1$ has an exceptional rôle in (27)), we see: (27) holds if $x_i \neq 0$ for $i = 2, \cdots, n$. For $x_1 \neq 0$ (27) held anyhow, and for some $i = 1, \cdots, n$ we must have $x_i \neq 0$. Therefore:

(27) *holds for all points* a: $[x_1 : \cdots : x_n]_r$.

8. Consider now two points a: $[x_1 : \cdots : x_n]_r$ and b: $[\xi_1 : \cdots : \xi_n]_r$. Put a': $[y_1 : \cdots : y_n]_l$, then $b \leqq a'$ means, considering (1) and (27) (cf. the end of 7.):

$$\sum_{i=1}^{n} w(x_i)\gamma_i\xi_i = 0. \tag{28}$$

$a \leqq a'$ can never hold ($a \cap a' = 0$, $a \neq 0$), so (28) can only hold for $x_i = \xi_i$, if all $x_i = 0$. Thus,

$$\sum_{i=1}^{n} w(x_i)\gamma_i x_i = 0 \text{ implies } x_1 = \cdots = x_n = 0. \tag{29}$$

Summing up the last result of 6., and formulae (26), (29) and (28), we obtain: *There exists an involutory antisomorphism* $w(x)$ *of* F (*cf.* (22), (25), (19)) *and a definite diagonal Hermitian form* $\sum_{i=1}^{n} w(x_i)\gamma_i\xi_i$ *in* F (*cf.* (26), (29)), *such that for* a: $[x_1 : \cdots : x_n]_r$, b: $[\xi_1 : \cdots : \xi_n]_r$ $b \leqq a'$ *is defined by polarity with respect to it:*

$$\sum_{i=1}^{n} w(x_i)\gamma_i\xi_i = 0. \tag{28}$$

This is exactly the result of §14, which is thus justified.

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CARDINAL NUMBERS ASSOCIATED WITH A POINT IN AN ABSTRACT SPACE*

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1. **Introduction.** The fundamental space of this paper, is a Fréchet V-space subject to the further restriction that there are no distinct points p and q such that q is a point of every neighborhood of p. Our understanding is that the Hausdorff Axiom A holds for every V-space, and that every set which contains a neighborhood of p is itself a neighborhood of p. The topological spaces of Hausdorff[1] are not sufficiently general for our purposes, since Theorem 13 states that Axiom C holds when certain other properties are present.

Sections 4, 5, 6, and 7 are devoted to showing that many theorems which were first proved under the hypothesis that the first enumerability axiom holds, are true more generally for spaces in which every point has a monotone complete family[2] of neighborhoods. The knowledge of such theorems led to a search for conditions that would imply the existence of monotone complete families of neighborhoods. It is shown in Theorem 1 that a necessary and sufficient condition for a point p to have such a family of neighborhoods, is the equality of two cardinal numbers, which for spaces satisfying the first enumerability axiom cannot exceed $\aleph_0$. In Section 3 there are various inequalities involving the six cardinal numbers considered in this paper.

2. **Definition of the cardinals.** The least cardinal which is the power of a complete family of neighborhoods of p is designated by $\chi(p)$, the least cardinal which is the power of a monotone family of neighborhoods whose product is p by $\psi'(p)$, the least cardinal which is the power of any family of neighborhoods whose product is p by $\psi(p)$, and the least cardinal which is the power of a family of neighborhoods of p whose product is not a neighborhood of p by $\eta(p)$. A set X is said to *converge to p with respect to power*, if for every neighborhood V of

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[1] A *Hausdorff space* is one satisfying the four axioms: A—*Every point p has a neighborhood and every neighborhood of p contains p*, B—*The product of two neighborhoods of p is a neighborhood of p*, C—*The interior of every set is an open set*, D—*Any two points have neighborhoods respectively which are disjoined.* Cf. Hausdorff, *Grundzüge der Mengenlehre*, first edition (Leipzig, 1914), p. 213. A Hausdorff space need not satisfy the First Enumerability Axiom of Hausdorff: $\chi(p) \leqq \aleph_0$ for every point p.

[2] A *complete family F of neighborhoods of p* is one such that any neighborhood of p contains as a subset a set of F. A *monotone family* is one such that of any two sets of the family, one is a subset of the other.

p, $|(X - p)\cdot V| > |X\cdot C(V)|$.[3] Then $\kappa(p)$[4] is the least cardinal which is the power of a set which converges to p with respect to power, and $\lambda(p)$ is the power of the least set X such that every neighborhood of p contains a point of $X - p$. For an isolated point p, $\chi(p) = \psi'(p) = \psi(p) = \eta(p) = 1$, while $\kappa(p)$ and $\lambda(p)$ are not defined. For non-isolated points p, $\chi(p)$, $\lambda(p)$, $\psi(p)$, and $\eta(p)$ are always defined. The only possible finite value of $\eta(p)$ is two, and a necessary and sufficient condition for the Hausdorff Axiom B is that $\eta(p) = 2$ for no point p. If any of the other five cardinals have finite values for the non-isolated point p, $\chi(p) = \psi(p)$, $\kappa(p)$ and $\psi'(p)$ do not exist, and $\eta(p) = 2$.

3. **Relations among the cardinals.** The inequalities, $\kappa(p) \geqq \lambda(p)$, $\chi(p) \geqq \psi'(p) \geqq \psi(p) \geqq \eta(p)$ and $\chi(p) \geqq \lambda(p)$, hold for all points p, if we leave out of consideration such of the cardinals as may fail to be defined at the point p.

THEOREM 1. *Suppose $\eta(p)$ exists and does not equal two. Then the following conditions are equivalent, and if they hold the six cardinals exist and are equal and this cardinal is regular.*[5]

(1). $\eta(p) = \chi(p)$.

(2). *There is a monotone complete family of neighborhoods of p.*

(3). *There is a complete family F, $|F|$ regular, of neighborhoods of p such that every subfamily of power $|F|$*[6] *is also a complete family.*

PROOF. Assume (2). The given monotone complete family G has a subfamily F consisting of sets $W_\alpha (\alpha < \Omega_{|F|}$, the least ordinal number of power $|F|)$ such that $W_{\alpha'}$ contains W_α if $\alpha < \alpha' < \Omega_{|F|}$, and such that any set of the original family contains a set of F.

Let the power of F be the minimum possible for such a subfamily. Since $\Omega_{|F|}$ is the least ordinal of power $|F|$, any subfamily $F' = [W_{\alpha\beta}]$ of F, $|F'| = |F|$, will also be of order type $\Omega_{|F|}$. Then F' is a complete family of neighborhoods of p, since for any W_α there is a $W_{\alpha\beta}$ such that $\alpha < \alpha_\beta < \Omega_{|F|}$, so that $W_\alpha \supset W_{\alpha\beta}$. Also $|F|$ is regular. If not, $|F|$ is the sum of γ cardinals μ_δ where $|F|$ exceeds γ and exceeds each μ_δ. Then the sequence $W_{\Omega_{\mu_\delta}}$ is a complete family of neighborhoods of p, contrary to the assumption that $|F|$ is the minimum power for a complete subfamily of G. To prove conversely that (3) implies (2), let W_1 be

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[3] $|X|$ denotes the cardinal number of the set X, and $C(V) = P - V$ the subset of the space P consisting of points not in V.

[4] $\kappa(p)$, $\chi(p)$ and $\psi(p)$ were introduced by P. Alexandroff and $\eta(p)$ by J. v. Neumann. See Alexandroff, Mathematische Annalen, Vol. 92 (1924), p. 267; Alexandroff and Urysohn, *Memoire sur les Espaces Compacts*, Verhandlungen Kon. A. Amsterdam, 1928; v. Neumann, Transactions of the American Mathematical Society, Vol. 37, p. 4. It was found by v. Neumann that his work on complete linear spaces could be extended from spaces satisfying the first enumerability axiom to those in which $\eta(p) = \psi(p)$.

[5] An *irregular cardinal* μ is one for which there exists a set of ν cardinals μ_α such that $\mu = \Sigma\mu_\alpha$, $\mu > \mu_\alpha$ for every μ_α and $\mu > \nu$. The least cardinal ν for which this is true is called the *least regular cardinal associated with* μ.

[6] $|F|$ designates the power or cardinal number of F.

any set of the family F of (3) and W_α any set of F contained in $\prod_{\alpha' \leqq \alpha} W_{\alpha'}$. The set W_α exists for each $\alpha < \Omega_{|F|}$, since $|F|$ is regular and therefore exceeds $\sum_{\alpha' < \alpha} |F_{\alpha'}|$ where $F_{\alpha'}$ is the subfamily consisting of all sets of F not contained in $W_{\alpha'}$. The monotone sequence W_α is a subfamily of F of power $|F|$, and is therefore a complete family of neighborhoods of p. To prove (1) implies (2), arrange the sets of a complete family of power χ in a sequence V_α of order type Ω_χ. For every $\alpha < \Omega_\chi$, the set $W_\alpha = \prod_{\alpha' \leqq \alpha} V_{\alpha'}$ will be a neighborhood of p since it is the product of neighborhoods which are less than η in number. Then W_α is the desired monotone complete family. Given, conversely, a monotone complete family, we can select as before a sub-family $F = [W_\alpha]$ of regular power and order type $\Omega_{|F|}$. Then $|F| \geqq \chi$ since F is complete, so it suffices to prove $|F| \leqq \eta$. Consider a family G of neighborhoods U of p, $|G| < |F|$. Since F is complete, each set U contains a set $W_{\alpha(U)}$ of F, and $\Pi U \supset \Pi W_{\alpha(U)}$. Since $|G| < |F|$ and $|F|$ is regular, there is a γ less than $\Omega_{|F|}$ such that $W_\gamma \subset \Pi W_{\alpha(U)} \subset \Pi U$. Therefore $|F| \geqq \eta$. For each α let x_α be a point of $W_\alpha - W_{\alpha+1}$. Then $X = [x_\alpha]$ converges to p with respect to power and κ exists. The existence of ψ' follows form the existence of a monotone complete family of neighborhoods of p, since we assumed that no point other than p is in all neighborhoods of p. Clearly ψ' is always a regular cardinal. So likewise is η, since from any family of power η of neighborhoods of p one can by taking products form a monotone family of neighborhoods whose product is not a neighborhood.

Theorem 1 is still true if in property (3) the only subfamilies of F which are required to be complete are those formed by omitting from F less than $|F|$ sets. It is not known whether or not there can be a family F of irregular power such that all subfamilies of power $|F|$ are complete, but if so it can be shown that η is the least regular cardinal associated with $|F|$.

THEOREM 2. *If $\chi(p) = \psi(p)$, $\kappa(p)$ exists and does not exceed $\psi(p)$, and if also $\psi(p)$ is irregular $\kappa(p)$ does not exceed the least regular cardinal associated with $\psi(p)$.*

This theorem is suggested by theorems 1 and 2 of Alexandroff. It contradicts his statement that the equality or inequality of ψ and χ is independent of the existence or non-existence of κ. The proof is made by arranging the sets of a complete family of power ψ of neighborhoods of p in a sequence V_α of order type Ω_ψ. Let $W_\alpha = \prod_{\beta \leqq \alpha} V_\beta$. Then no W_α is vacuous but two sets W_α may coincide. Let α' be any ordinal less than Ω_ψ such that $W_{\alpha'} \neq W_\alpha$ for every α less than α'. Let ξ be the order type of the α''s. For any point q other than p, there is a $V_{\alpha(q)}$, therefore a $W_{\alpha(q)}$, which does not contain q. If γ is the least α' such that $W_\gamma = W_{\alpha(q)}$, W_γ is a set $W_{\alpha'}$ which does not contain q. Then $\Pi V_{\alpha'} = p$, therefore there are ψ sets $W_{\alpha'}$ and ξ is coterminal with Ω_ψ. Let ν be the least cardinal such that there is a set of ordinals α_β of order type Ω_ν such that every α_β is an α' and such that for any α there is a β for which $\alpha < \alpha' < \alpha_\beta < \Omega_\nu$. Then $\nu = \psi$ if ψ is regular, otherwise ν is the least regular cardinal associated with ψ. Let x_β be a point of $W_{\alpha_\beta} - W_{\alpha_{\beta+1}}$. The set $X = [x_\beta]$ converges to p with respect to power, since any neighborhood V_α contains a set W_{α_β} and $|X - V_\alpha| \leqq |X - W_{\alpha_\beta}| < |X \cdot W_{\alpha_\beta}| \leqq |X \cdot V_\alpha|$.

The following theorem is analogous to the equivalence of properties (1) and (3) of Theorem 1, $\lambda(p)$ playing the rôle previously taken by $\eta(p)$.

THEOREM 3. *If there is a complete family F of regular power such that the product of the sets of each subfamily of power $|F|$ is p, $|F| = \chi(p) = \lambda(p)$. If $|F|$ is irregular, $\lambda(p)$ is not less than the least regular cardinal associated with $|F|$.*

PROOF. Let p be a point of accumulation but not an element of the set $X = [x]$. Let F_x be the subfamily consisting of the sets of F which contain x. Then $|F_x| < |F|$ for every x. Since $|F| = \sum_{x \in X} F_x$, λ equals $|F|$ if $|F|$ is regular and is not less than the least regular cardinal associated with $|F|$ if $|F|$ is irregular.

THEOREM 4. *If the family F of Theorem 3 exists and $|F| = \psi(p)$, $\kappa(p)$ exists and is regular and equals $\lambda(p)$. If $|F|$ is regular*

$$|F| = \chi(p) = \psi(p) = \kappa(p) = \lambda(p),$$

while if $|F|$ is irregular $\kappa(p) = \lambda(p)$ is the least regular cardinal associated with $|F| = \chi(p) = \psi(p)$.

Theorem 4 is an immediate consequence of Theorems 2 and 3.

LEMMA. *If a set X converges to p with respect to power, $\psi(p)$ is not less than $|X|$ or the least regular cardinal associated with $|X|$ according to whether $|X|$ is regular or irregular.*

THEOREM 5. *If F is the complete family of neighborhoods of p of regular power of Theorem 3, a necessary and sufficient condition that $|F| = \chi(p) = \psi(p)$ is that $\kappa(p)$ exists and is regular.*

PROOF. Theorem 5 follows directly from Theorem 4 and the Lemma. Consider a set X which converges to p with respect to power and a family F of power ψ of neighborhoods V of p such that $\Pi V = p$. Then $|X| = \Sigma |X \cdot C(V)|$ and $|X| > |X \cdot C(V)|$ for each V. The lemma follows immediately. It was proved by Alexandroff and Urysohn (loc. cit.) for $|X|$ regular.

Properties (1) and (2) of Theorem 1 are equivalent to the property, for any family F of power χ of neighborhoods of p there is a monotone family H of neighborhoods such that each set of F contains a set of H. These three properties are analogous respectively to the three properties of Theorem 6 below.

THEOREM 6. *If $\psi(p) = \eta(p)$, $\psi'(p)$ exists and equals $\psi(p)$. A necessary and sufficient condition that $\psi(p) = \eta(p)$ for a non-isolated point p, is that for any family F of $\psi(p)$ neighborhoods of p there is a monotone family H of neighborhoods such that each set of F contains a set of H.*

The theorem holds if H is replaced by a descending sequence S of order type Ω_ψ the product of whose sets is p. If $\psi = \eta$ there is a sequence V_α of order type Ω_η of neighborhoods of p such that $\Pi V_\alpha = p$. The existence of ψ' is established by considering the monotone sequence of sets $W_\alpha = \Pi_{\beta \leq \alpha} V_\beta$. The existence of the monotone family H is likewise established by taking products, but now this process is applied to the sets of the family G' consisting of all sets of the given family F and also all sets of some family G of ψ neighborhoods of p whose product is p. To prove conversely that $\psi = \eta$ we show that if $|F| < \eta$ the product of

the neighborhoods which constitute F is also a neighborhood. Form the family G' consisting of sets of F and sets of a family G selected as before. By hypothesis there is a monotone family H whose sets are contained in all sets of G'. For each set V of F choose a set of H contained in V and call the resulting subfamily H_F. Then there is a set of H contained in the product of the sets of H_F, therefore in the product of the sets of F. Otherwise, there would be a set of H_F in each set of H, and hence H_F would be a family of power less than ψ of neighborhoods of p whose product is p.

Theorem 7 below is obviously true, while Theorem 8 is an immediate consequence of Theorems 7 and 2 and the lemma which precedes Theorem 5.

THEOREM 7. *If $\psi'(p)$ exists, $\psi'(p) \leqq \lambda(p)$.*

THEOREM 8. *If $\psi'(p)$ and $\kappa(p)$ exist and the latter is regular,*

$$\lambda(p) = \kappa(p) = \psi(p) = \psi'(p).$$

If $\psi(p) = \chi(p)$ and $\psi'(p)$ exists, $\lambda(p) = \kappa(p) = \psi(p) = \psi'(p) = \chi(p)$, and this cardinal is regular.

If there exist sets X of regular cardinal number which converge to p with respect to power, $\lambda \leqq |X| \leqq \psi$, hence all such sets X are of power λ if ψ' exists or if the conditions of Theorem 5 hold. If there are monotone families of neighborhoods of p the product of which is p, all such families which have no subfamilies of lower power with the same properties, will be of power not greater than λ. Hence all such families will have the same cardinal number if κ exists and is regular.

4. **An equivalence.** I proved[7] for a space satisfying the first three axioms of Riesz and the first enumerability axiom, that the L-spaces are exactly those in which the fourth axiom of Riesz holds and are also exactly those in which property D of Hausdorff is valid. In Theorem 9, the first enumerability axiom ($\chi(p) = \eta(p) = \aleph_0$ for all points p) is replaced by condition $\chi(p) = \eta(p)$ for all points p. Since an L-space is one in which the topology can be defined by the convergence of enumerable sequences, it has $\eta(p) = \lambda(p) = \kappa(p) = \aleph_0$ for every non-isolated point p. Hence in considering spaces for which $\chi(p) = \eta(p)$, it is necessary to generalize the L-space, and this is done in two ways in properties III and III′. It is also desirable to consider besides the fourth axiom of Riesz, Property I, a slightly stronger property, I′.

I. Any set E such that E' (the set of all limit points or points of accumulation of E) contains the points p and q, has a subset X such that X' contains one but not both of the points p and q.

I′. As set E such that $E' \supset p + q$, has subsets X and Y such that X' contains p but not q and Y' contains q but not p.

II. Any two points have neighborhoods without common points.

III. A set can converge with respect to power to at most one point.

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[7] Bulletin of the American Mathematical Society, Vol. 37 (1931), p. 628.

III′. A set E having p as a limit point, has a subset X such that $X' = p$.

In any V-space II implies III and III′ implies I′ which implies I. If $\eta(p) = 2$ for no point p, II implies I′.

THEOREM 9. *If $\chi(p) = \eta(p) \neq 2$ for every non-isolated point p and has the same value for all such points, all the above properties are equivalent.*

PROOF. Except for III′, the theorem can be stated and proved for two fixed points p and q. If the points p and q of E' are contained respectively in disjoined sets V and W, p and q are respectively limit points of $V \cdot E$ and $W \cdot E$. Conversely, if there are no disjoined neighborhoods of p and q respectively, we shall exhibit a set $X = [x_\alpha]$ which converges to both p and q with respect to power and is such that any subset of X which has one of the two points as a limit point has the other also as a limit point. Let V_α and W_α be monotone complete families of neighborhoods of p and q respectively of order type Ω_χ. Let x_α be any point of $V_\alpha \cdot W_\alpha$. To show that II implies III′, consider the above family V_α of neighborhoods of p and select a point x_α in each V_α. Any other point q is not a limit point of X. For there will be neighborhoods V_α and W of p and q which are disjoined. Then q is not a limit point of the set $X \cdot V_\alpha$ which has no point in W, and is not a limit point of $X - V_\alpha$ which is a set of power less than $\chi = \lambda$, hence q is not a limit point of X.

5. **A sufficient condition for the first enumerability axiom.** For an L-space P satisfying axiom C of Hausdorff, a necessary and sufficient condition that every monotone descending sequence of closed sets be enumerable is that P be self-condensed.[8] This theorem was proved by Sierpinski and generalized by R. G. Putnam and by myself.[9] Substituting any regular cardinal μ for $\aleph_0$, a necessary and sufficient condition that P be μ-self-condensed is that every descending sequence of closed sets be of power less than μ. Furthermore, it is not necessary that P be an L-space or that axiom C hold. But if axiom C does not hold, one replaces sequences of closed sets by monotone sequences such that for two sets V and W of the sequence, $V \supset W$ implies that there is a point of V which is not in $\overline{W} = W + W'$. Suppose that $\chi(p) = \psi(p)$ is a regular cardinal and that for any other point x there are disjoined neighborhoods of p and x. Consider a sequence V_α of order type Ω_ψ of neighborhoods of p. Let $W_\alpha = \Pi_{\alpha' \leqq \alpha} V_{\alpha'}$. Select as follows from the sets W_α, a subsequence of distinct sets $W_{\alpha\beta}$ of order type Ω_ψ such that $W_{\alpha\beta}$ contains a point x_β not in $W_{\alpha\beta+1}$. Let x_β be any point in $W_{\alpha\beta}$, and let W and U be disjoined neighborhoods of x and p respectively, and let $\alpha_{\beta+1}$ be an ordinal such that $U \supset V_{\alpha\beta+1} \supset \overline{W}_{\alpha\beta+1}$. This ordinal exists since V_α is a complete family. For any limit ordinal β, let

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[8] A space P is *self-condensed* if every non-enumerable subset X of P *contains* a point x such that $| X \cdot U |$ is non-enumerable for every neighborhood U of x. The point x is a *condensation point* of X, whereas if $| X \cdot U | = | X |$, x would be a *nuclear point of* X.

[9] W. Sierpinski, Fundamenta Mathematica, Vol. 2 (1921), p. 179; Putnam, Bulletin American Mathematical Society, Vol. 36 (1930), p. 653; Robinson, loc. cit., p. 629, and American Journal of Mathematics, Vol. 55, p. 433.

α_β be an α such that $W_{\alpha\beta} \subset \Pi_{\gamma<\alpha\beta} W_\gamma$. But if P is μ-self-condensed the powers of the sequence $W_{\alpha\beta}$, which equals $\chi(p)$ and $\psi(p)$, will be less than μ, and the following theorem is proved.

THEOREM 10. *If a self-condensed space P satisfying axiom* D *of Hausdorff, has $\chi(p)$ equal to $\psi(p)$ and either regular or finite for every point p, P satisfies the first enumerability axiom. More generally, if P is μ-self-condensed and satisfies axiom* D, *$\chi(p) < \mu$ at points p for which $\chi(p)$ equals $\psi(p)$ and is regular.*

Obviously, a space satisfies the first enumerability axiom if it has an everywhere dense subset which is enumerable, and $\chi(p)$ equals $\eta(p)$ or $\lambda(p)$ at every point p.

6. **Regularity.** A *regular* Hausdorff space is a Hausdorff space in which if E is a closed set and p a point not in E, there are disjoined open sets containing E and p respectively. We define a *regular* V-space as one in which $\bar{E} \not\ni p$ implies that there are disjoined sets to which E and p are interior. An equivalent definition is, a space in which if p is interior to V there is also a set W to which p is interior such that $\overline{W} \subset V$. Any point p which has this property is valid is called a *regular point*, regardless of whether the space is regular.

Our modification of the definition of regularity, is due to the fact that closed sets do not play the same rôle in general V-spaces as in spaces satisfying axiom C, and it is analogous to our modification in Section 5 of the property concerning the power of monotone descending sequences of closed sets. However, the definition of regularity given above for Hausdorff spaces, can be applied to general V-spaces, and it yields a property which is weaker than regularity. It may be regarded as regularity in a new space (P, O) satisfying axiom C, the neighborhoods in (P, O) being only the sets which were open in the original space P.[10] It is equivalent to the property for any open set V and point p of V, there is an open set W and a closed set Z such that $V \supset Z \supset W \supset p$. A number of non-equivalent definitions of regularity could be secured by making V or W or both neighborhoods instead of open sets, and either using the closed set Z or the condition $\overline{W} \subset V$.

A locally compact regular Hausdorff space satisfies the first enumerability axiom. From that and another theorem of Alexandroff and Urysohn (loc. cit., p. 66) it follows that, given a locally compact Hausdorff space P in which $\psi(p) \leqq \aleph_0$ for all points, a necessary and sufficient condition that P is regular is that it satisfy the first enumerability axiom. The three following theorems generalize these results and make use of new properties $C^{\mu\nu}$ and $C_{\mu\nu}$.

A family F of sets is said to be a *proper covering* of the set V if every point of V is interior to some set of F, a *simple covering* if every point of V is contained in some set of F. The family F is *reducible* to a proper or a simple covering of V of power ν, if there is subfamily of F of power ν which constitutes a proper or a

[10] E. W. Chittenden, Transactions of the American Mathematical Society, Vol. 31 (1929), p. 296.

simple covering of V. Let $\mu(p)$ and $\nu(p)$ be functions of the point p whose values are cardinal numbers. If p has a neighborhood V such that every proper covering of P of power $\mu(p)$ is reducible to a proper (simple) covering of V of power less than $\nu(p)$, p is said to have property $C^{\mu\nu}$ ($C_{\mu\nu}$). A *space* has property $C^{\mu\nu}$ ($C_{\mu\nu}$) if each point p has *property* $C^{\mu\nu}$ ($C_{\mu\nu}$), the cardinals μ and ν being in general different for different points.

THEOREM 11. *If $\eta(p) \neq 2$, p has property $C^{\chi\eta}$, and for every other point q there are disjoined neighborhoods of p and q; p is a regular point. A space P which has property $C^{\chi\eta}$ and properties* B *and* D *of Hausdorff, is regular.*

PROOF. Let the sets V_α constitute a complete family of neighborhoods of q of power $\chi(p)$. For each point q of $\bar{V}\cdot\bar{E}$, let V_α and W_q be disjoined neighborhoods of p and q respectively. For each α, form a new set W_α by lumping together all sets W_q which contain no points of V_α. These sets W_α together with $C(E)$ and $C(V)$ form a family F of power $\chi(p)$ which properly covers P, hence has a subfamily F' of power less than $\eta(p)$ (consisting of $C(E)$ and certain sets W_α which we designate by $W_{\alpha\beta}$) which properly covers V. The product $V\cdot\Pi\, V_{\alpha\beta}$ is the desired neighborhood of p, while the desired set to which E is interior is $C(V) + \Sigma\, W_{\alpha\beta}$.

THEOREM 12. *If $\psi'(p)$ exists, $\eta(p) \neq 2$, and the regular point p has property $C_{\psi'\psi'}$, then $\chi(p) = \psi'(p)$. If a regular space P has properties* B *and $C_{\psi'\psi'}$, and $\psi'(p)$ exists for all points p, then $\chi(p) = \psi'(p)$ at all points p.*

PROOF. Let W_α be a monotone descending sequence of neighborhoods of p of order type $\Omega_{\psi'}$, $\Pi\, W_\alpha = p$. Since p is a regular point and $\eta(p) \neq 2$, we may assume that $\overline{W}_{\alpha+1} \subset W_\alpha \subset V$. To show that the sets W_α constitute a complete family, we prove that an arbitrary neighborhood U of p contains a set W_α. The family F of power $\psi'(p)$ composed of U and the sets $C(W_\alpha)$ properly covers P, hence has a subfamily F' of lower power which simply covers V. Then F' simply covers P, since $C(V) \subset C(W_\alpha)$ for every α. As $|F'| < \psi'(p)$ and $\psi'(p)$ is regular, there is a fixed ordinal γ such that $\alpha < \gamma < \Omega_{\psi'}$ for every α such that $C(W_\alpha)$ is a set of F'. Therefore the set $U + C(W_\gamma)$ contains all points of P, and U contains W_γ. This theorem is clearly true if one substitutes $|F|$ for $\psi'(p)$, where $|F|$ is any regular cardinal for which there exists a monotone descending sequence of order type $\Omega_{|F|}$ of neighborhoods of p whose product is p. Furthermore, the hypothesis $\eta(p) \neq 2$ could be omitted if one assumed that $V = P$. Finally, the first corollary below states a stronger result which holds only when there is reducibility to finite coverings. The other corollaries follow immediately from Theorems 11 and 12. The three corollaries also hold for single points.

COROLLARY. *In a space which has properties* B *and* D *of Hausdorff and property $C_{\psi\aleph_0}$, $\chi(p) = \psi(p)$ for each point p.*

COROLLARY. *If a space has properties* B *and* D *and $C^{\chi\eta}$ and $\psi'(p)$ exists for each point p, $\chi(p) = \psi'(p)$ for each point p.*

COROLLARY. *If a space P has properties $C^{\psi\psi}$ and* B *and $\psi(p) = \eta(p)$ at each point, a necessary and sufficient condition that P be regular, is that property* D *hold and $\chi(p) = \psi(p)$ at each point p.*

7. **Axiom** C. In a previous paper we proved that a locally bicompact space which has properties B and D also has property C.[11] We stated that the theorems of that paper hold, if one replaces the hypothesis of reducibility of coverings of all infinite powers by the first enumerability axiom and the reducibility of enumerable coverings. The latter theorems can be generalized as shown by the two examples which follow. The proofs are like that of Theorem 11.

THEOREM 13. *If a space P has property* D *and each non-isolated point p has a neighborhood V such that every proper covering of P of power $\chi(p)$ is reducible to a proper covering of V of power less than $\eta(p)$, P is a regular Hausdorff space.*

THEOREM 14. *A set E is closed, if for each point p of C(E) every proper covering of E of power $\chi(p)$ is reducible to a simple covering of power less than $\eta(p)$.*

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[11] Bulletin of the American Mathematical Society, Vol. 39 (1933), p. 599.

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ELEMENTARY PROOF OF THE SPECTRAL THEOREM

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1. **Introduction.** Proofs of the fundamental theorem on the spectral resolution of self-adjoint operators in Hilbert space have been given by Carleman [1], Hellinger [2], Hilbert [3], Koopman and Doob [4], von Neumann [5], Riesz [6], Stone [7], Wecken [8], and Wintner [9], using a great variety of methods. Some of these methods apply only to bounded operators, while others are suited to the general case. The resolution for bounded operators yields the resolution for non-bounded operators as indicated by von Neumann [5], Riesz [6], and Riesz and Lorch [10]. Every proof of the fundamental theorem so far devised makes use of some part of the theory of functions. Thus Carleman, Hilbert, Stone, and Wintner employ the selection principle of Helly to pass by approximation from the familiar case of finite Hermitean matrices to the general case. Carleman, Hellinger, Koopman and Doob, and Stone use facts about the representation of analytic functions by Stieltjes integrals. The procedures of von Neumann and Riesz repose upon the Weierstrass approximation theorem. Finally, the proof of Wecken involves the theory of the power series representation of the function $\sqrt{(1-x)}$.

We shall now give a new proof of the spectral theorem which is strictly elementary in the sense that it depends only upon *intrinsic* properties of Hilbert space and of the real number system. Our method may be regarded as a variation on the extremal principles devised by Schmidt [11] in the theory of integral equations and applicable to the case of totally continuous self-adjoint operators in Hilbert space. It is well known that these extremal principles do not suffice to resolve even bounded self-adjoint operators. Hence it is not to be expected that our method will exhibit a very close technical relation to Schmidt's.

In the present paper we shall be concerned primarily with bounded self-adjoint operators. Indeed our method, which requires the study of the powers of an operator, is not suited to the case of non-bounded operators. To discuss that case we must appeal to other devices. In a later paper, one of us (Stone) proposes to give a new treatment of several branches of operator theory starting from the results given explicitly in these pages. In particular, the entire spectral theory will be developed on this basis. For the purposes of this projected paper, the Stieltjes integral representation of a bounded self-adjoint operator may be regarded as a digression, since all the information which it provides may be deduced equally well from the approximation theorem given in the last section below.

Since the dimensionality postulates occurring in the definition of Hilbert space are actually so little used in the theory, it is convenient to suppress them. Thus

we shall consider those spaces which are called, following Löwig [12], complex Euclidean spaces.

2. **Complex Euclidean Space.** A complex Euclidean space $\mathfrak{L}$ is defined as a class of abstract elements subject to the postulates A, B, and E of von Neumann [5, 7]; that is to say, it is a complex vector space or module which possesses a metric derived from a complex-valued Hermitian symmetric bilinear form or inner product and which is complete with respect to metric convergence. If $\mathfrak{L}$ has a finite basis with respect to linear combination, it is said to be a unitary space [7]; and, if $\mathfrak{L}$ has no such basis but is separable, it is said to be a Hilbert space [5, 7]. The geometry of $\mathfrak{L}$ is essentially independent of such restrictions on its dimensionality; the cardinal number of a complete orthonormal set in $\mathfrak{L}$ may, however, exceed $\aleph_0$. The elementary parts of operator-theory are likewise essentially independent of dimensionality restrictions. In this connection, it is important to note that the Fréchet-Riesz theorem in Hilbert space, according to which a continuous linear functional $L(f)$ necessarily has the form $L(f) = (f, g)$, has been extended to complex Euclidean space by Löwig [12] and Riesz [13].

We shall find use for two sorts of convergence in a space $\mathfrak{L}$—namely, *strong* convergence and *weak* convergence. A sequence $\{f_n\}$ is said to converge strongly to the limit f if $|f - f_n|$ tends to zero with $1/n$; and is said to converge weakly to the limit f if $(f - f_n, g) = (f, g) - (f_n, g)$ tends to zero with $1/n$ for every g in $\mathfrak{L}$. While the properties of these two types of convergence are generally known, we may recall the properties of weak convergence on which we rely in the sequel: (1) if $\{f_n\}$ converges weakly to f, then the sequence $\{|f_n|\}$ is bounded [14] and $|f| \leqq \liminf_{n\to\infty} |f_n|$; (2) if $\{|f_n|\}$ is bounded, then $\{f_n\}$ contains a subsequence converging weakly to a limit f; (3) if the sequence $\{f_n\}$ lies in a (strongly) closed linear manifold $\mathfrak{M}$ and converges weakly to f, then f lies in $\mathfrak{M}$; (4) in a unitary space, weak and strong convergence are equivalent; (5) if $\{f_n\}$ converges weakly to f and $\{g_n\}$ strongly to g, then (f_n, g_n) tends to (f, g).

We shall also require certain information concerning operator-sequences. The basic result reads as follows: if A_n, B_n, A, and B are linear operators defined over $\mathfrak{L}$ and subject to the conditions

$$(1)\ |A_n f| \leqq \alpha |f|; \qquad (2)\ A_n f \to Af \text{ strongly};$$

$$(3)\ |B_n f| \leqq \beta |f|; \qquad (4)\ B_n f \to Bf \text{ strongly};—$$

then the sequence $\{A_n B_n\}$ has the properties

$$(5)\ |A_n B_n f| \leqq \alpha\beta |f|; \qquad (6)\ A_n B_n f \to ABf \text{ strongly}.$$

A proof is easily supplied on the basis of the inequalities

$$|A_n B_n f| \leqq \alpha |B_n f| \leqq \alpha\beta |f|,$$

$$\begin{aligned} |ABf - A_n B_n f| &\leqq |A(Bf) - A_n(Bf)| + |A_n(Bf - B_n f)| \\ &\leqq |A(Bf) - A_n(Bf)| + \alpha |Bf - B_n f|. \end{aligned}$$

The extension of this result by induction to the case of any finite number of operator-sequences is evident.

3. **The Bounds of Linear Symmetric Operators.** It will be convenient to discuss in some detail the not unfamiliar facts concerning the bounds of linear symmetric operators. An operator A is said to be symmetric if its domain generates a closed linear manifold coincident with $\mathfrak{L}$ and if $(Af, g) = (f, Ag)$ for all f and g in its domain. Throughout this section we suppose that A is both symmetric and linear; but we do not require that it be continuous. For such an operator the inner product (Af, f) is real; and its greatest lower bound α and least upper bound β under the condition $|f| = 1$ are the same as the corresponding bounds of the quotient $(Af, f)/(f, f)$ where $f \neq 0$. We admit the cases $\alpha = -\infty$, $\beta = +\infty$ to consideration; and we point out explicitly that when $\mathfrak{L}$ consists of the null element alone the numbers α and β are not defined. We call α and β the bounds, lower and upper respectively, of the operator A. The following properties of the bounds are now evident:

THEOREM 1. *If A has bounds α and β, then $A + \lambda I$ has bounds $\alpha + \lambda$ and $\beta + \lambda$ for arbitrary real λ; if ρ is real and positive, ρA has bounds $\rho\alpha$ and $\rho\beta$; and $-A$ has bounds $-\beta$ and $-\alpha$.*

We recall also the well-known result:

THEOREM 2. *The least upper bounds of the quotients $(Af, g)/|f|\,|g|$ and $|Af|/|f|$, where $f \neq 0$ and $g \neq 0$, have* $\max(|\alpha|, |\beta|)$ *as their common value when A has bounds α and β.*

It will be useful in the sequel to know under what conditions the bounds of an operator are attained. We have:

THEOREM 3. *In order that a bound of A be attained, it is necessary and sufficient that it be a characteristic value of A; and the elements for which the bound is attained are precisely the corresponding characteristic elements of A.*

In proving this theorem, we first note that an infinite bound is never attained. Thus Theorem 1 permits us to reduce the consideration of an attained bound to the case of an operator A with attained lower bound $\alpha = 0$. We thus suppose that for all f in the domain of A we have $(Af, f) \geqq 0$, while for some g in that domain we have $(Ag, g) = 0$, $|g| = 1$. If l is an arbitrary complex number and h an arbitrary element of the domain of A, we put $f = g + lh$ in the inequality above and obtain the new inequality

$$\bar{l}(Ah, g) + l(Ag, h) + |l|^2 |h|^2 \geqq 0.$$

We now restrict l to have the form $l = \lambda(Ag, h)$, where λ is real. By virtue of the relations $(Ah, g) = (h, Ag) = \overline{(Ag, h)}$, our inequality becomes

$$|(Ag, h)|^2 \lambda(2 + \lambda |h|^2) \geqq 0.$$

Since λ is arbitrary, we must have $(Ag, h) = 0$; and since h is arbitrary in the domain of A, we must even have $Ag = 0$. Thus g is a characteristic element of A corresponding to the characteristic value zero. It is trivial that such an element is one for which the lower bound of A is attained.

If E is a projection permutable with A in the sense that, when Af exists, AEf also exists and is equal to EAf,—then A may be considered in terms of its behavior in the ranges $\mathfrak{M}$ and $\mathfrak{N}$ of E and $I-E$ respectively. It is a linear symmetric operator in each of these closed linear manifolds. It must be noted, of course, that the elements Ef, where f is in the domain of A, generate the closed linear manifold $\mathfrak{M}$: for if g is in $\mathfrak{M}$, then $(Ef, g) = 0$ implies $(f, g) = (f, Eg) = (Ef, g) = 0$ and hence $g = 0$. Consequently we may refer to the bounds of A in $\mathfrak{M}$ and in $\mathfrak{N}$ respectively. Since the domain of A consists of all elements $f + g$ where f and g are elements of the domain restricted to lie in $\mathfrak{M}$ and in $\mathfrak{N}$ respectively, and since

$$(A(f + g), f + g) = (Af, f) + (Ag, g), \; |f + g|^2 = |f|^2 + |g|^2,$$

we see that the bounds of A in $\mathfrak{L}$ are obtained by taking the maximum and the minimum of its bounds in $\mathfrak{M}$ and $\mathfrak{N}$ respectively.

4. **Spectral Theory for Bounded Definite Self-Adjoint Operators.** Our method of constructing the spectral theory is adapted to the case of self-adjoint operators—that is, the linear symmetric operators characterized by the identity $A \equiv A^*$—for which the bounds α and β satisfy the inequalities

$$0 \leqq \alpha \leqq \beta < +\infty.$$

Since the apparently more general case where α is restricted only to be finite is easily reduced by virtue of Theorem 1 to the one under consideration, we do not actually lose by the supposition that $\alpha \geqq 0$. The details of the general case will be discussed in the following section. We recall that a self-adjoint operator with finite bounds is defined over the entire space $\mathfrak{L}$.

We state the spectral theorem for the indicated case in the following terms:

THEOREM 4. *If H is a self-adjoint operator with finite upper bound and non-negative lower bound, then the class $\mathfrak{M}(\lambda)$, $-\infty < \lambda < +\infty$, of all elements in $\mathfrak{L}$ for which the condition*

$$|H^k f| \leqq \lambda^k |f| \text{ for } k \geqq 1, \tag{1}$$

or, when $\lambda > 0$, the equivalent condition

$$\limsup_{k\to\infty} |H^k f|/\lambda^k < +\infty, \tag{2}$$

is satisfied, is a closed linear manifold; its projection operator $F(\lambda)$ is permutable with H and also with every bounded linear operator which is permutable with H; and in $\mathfrak{M}(\lambda)$ the upper bound of H does not exceed λ. In the range $\mathfrak{N}(\lambda)$ of $I - F(\lambda)$ the lower bound of H is not less than λ; and, if it is equal to λ, is not attained. The inequality $\mu \leqq \lambda$ implies the equivalent relations $\mathfrak{M}(\mu) \subset \mathfrak{M}(\lambda)$, $F(\mu)F(\lambda) = F(\mu)$; the inequality $\lambda < 0$ implies the equivalent relations $\mathfrak{M}(\lambda) = \mathfrak{O}$, $F(\lambda) = O$; and the inequality $\lambda \geqq \beta$, where β is the upper bound of H, implies the equivalent relations $\mathfrak{M}(\lambda) = \mathfrak{L}$, $F(\lambda) = I$. Finally, when $\epsilon > 0$ tends to zero, $F(\lambda + \epsilon)f$ tends strongly to $F(\lambda)f$ for every f in $\mathfrak{L}$.

We shall break the proof of this theorem into several parts.

First let us dispose of the concluding portions of the theorem, which are all elementary consequences of condition (1). If $\lambda < 0$, the inequality $|Hf| \leqq \lambda |f|$ of the condition (1) implies $f = 0$; and we therefore have $\mathfrak{M}(\lambda) = \mathfrak{O}$. Thus in considering the inequality $\mu \leqq \lambda$ we may suppose further that $\mu \geqq 0$; and we then see that, if f lies in $\mathfrak{M}(\mu)$, the inequalities $|H^k f| \leqq \mu^k |f| \leqq \lambda^k |f|$ are valid for $k \geqq 1$ and imply that f lies also in $\mathfrak{M}(\lambda)$. If $\lambda \geqq \beta$, where β is the upper bound of H, we note that $|H^k f| \leqq \beta |H^{k-1} f| \leqq \lambda |H^{k-1} f|$ for $k \geqq 1$ and for every f in $\mathfrak{L}$ in accordance with Theorem 2; and we then easily conclude that $\mathfrak{M}(\lambda) = \mathfrak{L}$. Obviously, if $|H^k f| \leqq (\lambda + \epsilon)^k |f|$ for $k \geqq 1$ and for every $\epsilon > 0$, the element f has the property that $|H^k f| \leqq \lambda^k |f|$ for $k \geqq 1$; in other words, the manifold $\mathfrak{M}(\lambda)$ not only is included in all the manifolds $\mathfrak{M}(\lambda + \epsilon)$ for $\epsilon > 0$ but actually coincides with their intersection. On well known grounds [7], this geometrical property of the manifolds $\mathfrak{M}(\lambda)$ is equivalent to the property that the corresponding projections $F(\lambda)$ satisfy $F(\lambda + \epsilon)f \to F(\lambda)f$ strongly for all f in $\mathfrak{L}$ when ϵ tends to zero through positive values. It will be observed that the family of projections $F(\lambda)$ thus enjoys all the characteristic properties of a resolution of the identity [7].

We are thus left to prove the first two statements of the theorem. We show next that it is sufficient for us to consider the special case $\lambda = 1$. The case where $\lambda < 0$ is trivial since $\mathfrak{M}(\lambda) = \mathfrak{O}$, $F(\lambda) = O$ as we have already remarked. The case where $\lambda = 0$ is also trivial: in this case the condition (1) reduces to $Hf = 0$, so that $\mathfrak{M}(0)$ either is the characteristic manifold of H corresponding to the characteristic value 0 or again reduces to $\mathfrak{O}$. Thus the assertions of the theorem concerning $\mathfrak{M}(0)$ and $F(0)$ are easily verified. First, if T is a bounded linear operator defined over $\mathfrak{L}$ and permutable with H, then T^* has like properties since $T^*H = T^*H^* = (HT)^* = (TH)^* = H^*T^* = HT^*$; and T and T^* both transform $\mathfrak{M}(0)$ into a part of itself since $Hf = 0$ implies $HTf = THf = 0$, $HT^*f = T^*Hf = 0$. On well known grounds [7], this geometrical property of $\mathfrak{M}(0)$ and T is equivalent to the relation $TF(0) = F(0)T$. In particular, we may take $T = H$. Secondly, the lower bound of H in $\mathfrak{N}(0)$, the orthogonal complement of $\mathfrak{M}(0)$, cannot be less than the lower bound of H in $\mathfrak{L}$ and thus cannot be less than zero; and Theorem 3 shows that if this bound is equal to zero it cannot be attained. Having disposed in this manner of the cases $\lambda \leqq 0$, we turn to the case $\lambda > 0$. Here we see that the first two statements of Theorem 4 are unchanged in meaning and content if we write H/λ for H and everywhere else put $\lambda = 1$: for permutability with H/λ is equivalent to permutability with H, and the bounds of H/λ are related to those of H by Theorem 1. Since the correspondence $H/\lambda \leftrightarrow H$ does not alter the class of operator under consideration, we are thus enabled to confine our attention, when $\lambda > 0$, to the single special case $\lambda = 1$.

We are therefore left to prove the first two statements of Theorem 4 with $\lambda = 1$. We now separate these statements and prove them consecutively as Theorem 4a and Theorem 4b.

THEOREM 4a. *If H has the properties assumed above, then the class $\mathfrak{M}$ of all elements in $\mathfrak{L}$ for which the equivalent conditions*

$$|H^k f| \leqq |f| \text{ for } k \geqq 1, \tag{1}$$

$$\limsup_{k\to\infty} |H^k f| < +\infty, \tag{2}$$

are satisfied is a closed linear manifold; its projection operator F is permutable with H and also with every bounded linear operator which is permutable with H; and in $\mathfrak{M}$ the upper bound of H does not exceed 1. In the range $\mathfrak{N}$ of $I - F$ the relation $\limsup_{k\to\infty} |H^k f| = +\infty$ *holds whenever $f \neq 0$.*

If f is any element of $\mathfrak{L}$, then $\sigma(f) = \limsup_{k\to\infty} |H^k f|$ exists and satisfies the inequalities $0 \leqq \sigma(f) \leqq +\infty$. It is also evident that $\sigma(0) = 0$, $\sigma(af) = |a|\sigma(f)$ when $a \neq 0$, and $\sigma(f + g) \leqq \sigma(f) + \sigma(g)$. Hence the inequality $\sigma(f) < +\infty$ specifies a linear manifold $\mathfrak{M}$. If T is a bounded linear operator defined over $\mathfrak{L}$ and permutable with H, the relations $|Tf| \leqq \gamma |f|$, $|H^k Tf| = |TH^k f| \leqq \gamma |H^k f|$ show that $\sigma(Tf) \leqq \gamma\sigma(f)$. It follows that every such operator T transforms $\mathfrak{M}$ into a part of itself.

We now observe the relations

$$0 \leqq |H^k f|^2 = (H^k f, H^k f) = (H^{k-1} f, H^{k+1} f) \leqq |H^{k-1} f|\,|H^{k+1} f|,$$

holding for $k \geqq 1$ and for every f in $\mathfrak{L}$. They show at once that, if $|H^p f| = 0$ for some $p \geqq 2$, then also

$$|Hf| = |H^2 f| = \cdots = |H^p f| = 0.$$

Consequently we can throw the above inequalities into the form

$$|H^k f| / |H^{k-1} f| \leqq |H^{k+1} f| / |H^k f|, \qquad k \geqq 1,$$

whenever $Hf \neq 0$. They then show that the sequence $\{\tau_k(f)\}$, where $\tau_k(f) = |H^k f| / |H^{k-1} f|$, increases monotonely to a positive limit $\tau(f)$. When $Hf = 0$ we define $\tau(f) = 0$.

In order to establish the equivalence of conditions (1) and (2) we next investigate the relations between $\sigma(f)$ and $\tau(f)$. If $\tau(f) \leqq 1$, we have

$$|H^k f| / |f| = \tau_1(f) \cdots \tau_k(f) \leqq \tau^k(f) \leqq 1, \qquad |H^k f| \leqq |f|$$

for $k \geqq 1$. It follows that $\sigma(f) \leqq 1$ and hence that $\sigma(f) < +\infty$. If $\tau(f) > 1$, there exists a real number ρ such that $\tau(f) > \rho > 1$ and an integer p such that $\tau_k(f) > \rho$ for $k > p$. Thus we have

$$|H^{n+p} f| / |H^p f| = \tau_{p+1}(f) \cdots \tau_{n+p}(f) > \rho^n,$$

$$|H^{n+p} f| > \rho^n |H^p f|$$

for $n \geqq 1$. Since $H^p f \neq 0$, it follows that $\sigma(f) = +\infty$. Thus we see that the inequalities $\tau(f) \leqq 1$, $\tau(f) > 1$ are equivalent respectively to the relations $\sigma(f) < +\infty$, $\sigma(f) = +\infty$; and also that $\sigma(f) < +\infty$ implies $\sigma(f) \leqq 1$. Furthermore we see that $\sigma(f) < +\infty$ if and only if $|H^k f| \leqq |f|$ for $k \geqq 1$.

Since H^k is a continuous operator, the class of elements specified by the inequality $|H^kf| \leqq |f|$ is necessarily a closed subset of $\mathfrak{L}$. Hence the class specified by the simultaneous inequalities $|H^kf| \leqq |f|$, $k \geqq 1$, being the intersection of closed sets, is itself a closed subset of $\mathfrak{L}$. We thus conclude that the linear manifold $\mathfrak{M}$ is closed. Now we have noted above that a bounded linear operator T permutable with H transforms $\mathfrak{M}$ into a part of itself. Since T^* is then also permutable with H, we see that T^* likewise transforms $\mathfrak{M}$ into a part of itself. In consequence, the projection operator F of $\mathfrak{M}$ is permutable with T. In particular we may take $T = H$. Since by condition (1) the inequality $|Hf| \leqq |f|$ holds throughout $\mathfrak{M}$, we see by Theorem 2 that the upper bound of H in $\mathfrak{M}$ does not exceed 1.

The manifold $\mathfrak{N}$ described in the statement of the theorem is the orthogonal complement of $\mathfrak{M}$ and thus has no element except the null element in common with $\mathfrak{M}$. Hence, if f lies in $\mathfrak{N}$ and $f \neq 0$, we have $\sigma(f) = +\infty$.

We may remark that in the proof of Theorem 4a we have not made any essential use of the definiteness of H. This assumption comes into play in the following theorem.

THEOREM 4b. *With the assumptions and notations of Theorem* 4a, *the lower bound of H in $\mathfrak{N}$ is not less than* 1; *and, if equal to* 1, *is not attained. In other words, if* $\limsup_{k\to\infty} |H^kf| = +\infty$ *for every $f \neq 0$ in a space $\mathfrak{N}$, then $(Hf, f) > (f, f)$ for every such f.*

Clearly we may restrict our attention to the behavior of H in $\mathfrak{N}$. We know that $\limsup_{k\to\infty} |H^kf| = +\infty$ for every $f \neq 0$ in $\mathfrak{N}$. We denote by α the lower bound of H in $\mathfrak{N}$. By our assumptions on H we know that $\alpha \geqq 0$. We wish to prove that $\alpha \geqq 1$. In reaching this result, we must proceed by cases.

We begin with the case where $\mathfrak{N}$ is a unitary space. The trivial case where $\mathfrak{N}$ consists of the null element alone may be excluded. Then, by definition, there exists a sequence $\{f_n\}$ such that $(Hf_n, f_n) \to \alpha$, $|f_n| = 1$. Since the sequence $\{|f_n|\}$ is bounded, there exists a subsequence of $\{f_n\}$ which converges weakly to a limit f. The fact that $\mathfrak{N}$ is unitary implies that the convergence is strong as well as weak. Hence passage to the limit yields the relations $(Hf, f) = \alpha$, $|f| = 1$. Since the bound α is attained, we conclude that $Hf = \alpha f$ in accordance with Theorem 3. Thus $|H^kf| = \alpha^k |f|$ for $k \geqq 1$ so that $\sigma(f) = +\infty$ implies $\alpha > 1$, as we wished to prove.

We proceed next to the case where $\mathfrak{N}$ is a Hilbert space. If ϵ is a positive number, then by definition there exists an element φ_1 in $\mathfrak{N}$ such that $(H\varphi_1, \varphi_1) \leqq \alpha + \epsilon$, $|\varphi_1| = 1$. We form a complete orthonormal set $\{\varphi_n\}$ in $\mathfrak{N}$ with φ_1 as its first element. The projection of $\mathfrak{N}$ on the closed linear manifold generated by $\varphi_1, \cdots, \varphi_n$ will be denoted by E_n. We consider the self-adjoint operator $H_n = E_nHE_n/(\alpha + 2\epsilon)$. Since $(E_nHE_nf, f) = (HE_nf, E_nf) \geqq 0$ for all f in $\mathfrak{N}$, the lower bound of H_n is not less than zero. If β is the upper bound of H in $\mathfrak{N}$, we have $|E_nHE_nf| \leqq |HE_nf| \leqq \beta |E_nf| \leqq \beta |f|$ and hence

$$|H_nf| \leqq (\beta/(\alpha + 2\epsilon))\, |f|.$$

We can therefore apply Theorem 4a to the operator H_n, constructing the closed linear manifold $\mathfrak{M}_n$ specified by the conditions $|H_n^k f| \leqq |f|$, $k \geqq 1$, the orthogonal complement $\mathfrak{N}_n$, and the projection operator F_n, as described in the theorem. If f is in the range of the projection $I - E_n$, we have $H_n f = H_n(I - E_n)f = E_n H E_n(I - E_n)f/(\alpha + 2\epsilon) = 0$. Thus $\mathfrak{M}_n$ contains the range of $I - E_n$; and $\mathfrak{N}_n$ is contained in the range of E_n. Since the range of E_n is a unitary space by definition, the manifold $\mathfrak{N}_n$ is a unitary space also. It follows that the results of the preceding paragraph are applicable to H_n: in particular, the lower bound of H_n in $\mathfrak{N}_n$ is not less than 1. If we make use of the permutability of the projection F_n with H_n, we see that

$$(H_n(I - F_n)\varphi_1, (I - F_n)\varphi_1) = (H_n\varphi_1, \varphi_1) - (H_n F_n\varphi_1, F_n\varphi_1) \leqq (H_n\varphi_1, \varphi_1)$$

in accordance with the inequality $(H_n f, f) \geqq 0$. If we recall further that $E_n\varphi_1 = \varphi_1$, we see that

$$(E_n H E_n\varphi_1, \varphi_1) = (H E_n\varphi_1, E_n\varphi_1) = (H\varphi_1, \varphi_1),$$

$$(H_n\varphi_1, \varphi_1) = (H\varphi_1, \varphi_1)/(\alpha + 2\epsilon).$$

Thus we have

$$\begin{aligned} 1 - (F_n\varphi_1, \varphi_1) &= |(I - F_n)\varphi_1|^2 \leqq (H_n(I - F_n)\varphi_1, (I - F_n)\varphi_1) \\ &\leqq (H_n\varphi_1, \varphi_1) = (H\varphi_1, \varphi_1)/(\alpha + 2\epsilon) \\ &\leqq \frac{\alpha + \epsilon}{\alpha + 2\epsilon} \end{aligned}$$

by virtue of the fact that $(I - F_n)\varphi_1$ lies in $\mathfrak{N}_n$ and of our initial determination of φ_1. Finally we obtain the inequalities

$$0 < \frac{\epsilon}{\alpha + 2\epsilon} \leqq (F_n\varphi_1, \varphi_1) = |F_n\varphi_1|^2 \leqq |\varphi_1|^2 = 1$$

from those just established and from the fact that F_n is a projection. Thus the sequence $\{f_n\}$, where $f_n = F_n\varphi_1$, contains a subsequence converging weakly to a limit f in $\mathfrak{N}$; and this limit cannot be the null element, since

$$(f, \varphi_1) = \lim_{n\to\infty}(f_n, \varphi_1) \geqq \epsilon/(\alpha + 2\epsilon)$$

when n becomes infinite through the subscripts of this subsequence. We confine our attention henceforth to this subsequence, renumbering its elements and those of the associated operator sequences $\{E_n\}$, $\{F_n\}$, and $\{H_n\}$ accordingly. Since $E_n g$ tends strongly to g and $|E_n g| \leqq |g|$ for every g in $\mathfrak{N}$, we can apply the closing remarks of §2 to show that $H_n^k g$ tends strongly to $H^k g/(\alpha + 2\epsilon)^k$ for every g in $\mathfrak{N}$. It is now easy to show that $H_n^k f_n$ converges weakly to $H^k f/(\alpha + 2\epsilon)^k$: for we have

$$(H_n^k f_n, g) = (f_n, H_n^k g) \to (f, H^k g/(\alpha + 2\epsilon)^k) = (H^k f/(\alpha + 2\epsilon)^k, g)$$

for every g in $\mathfrak{R}$. Since $F_n\varphi_1$ lies in $\mathfrak{M}_n$, where the upper bound of H_n is not greater than 1, we see that

$$|H_n^k f_n| = |H_n^k F_n \varphi_1| \leqq |F_n \varphi_1| \leqq 1, \qquad k \geqq 1,$$

and hence, on passing to the limit, that

$$|H^k f/(\alpha + 2\epsilon)^k| \leqq 1, \qquad k \geqq 1.$$

Writing this inequality in the form $|H^k f| \leqq (\alpha + 2\epsilon)^k$ and noting that $f \neq 0$ implies $\sigma(f) = +\infty$, we conclude that $\alpha + 2\epsilon > 1$. Since ϵ was an arbitrary positive number, we must have $\alpha \geqq 1$, as we wished to prove.

We can now proceed to the case where $\mathfrak{R}$ is entirely unrestricted. Taking f as an arbitrary element in $\mathfrak{R}$, we form the closed linear manifold $\mathfrak{R}_f$ generated by the elements f, Hf, H^2f, $\cdots$. This manifold, being separable, is either a unitary space or a Hilbert space. Obviously H carries $\mathfrak{R}_f$ into a part of itself and is therefore permutable with the projection operator of $\mathfrak{R}_f$; and we may therefore confine our attention to the behavior of H in $\mathfrak{R}_f$. Since $\mathfrak{R}_f$ is contained in $\mathfrak{R}$, we have $\sigma(g) = +\infty$ for every $g \neq 0$ in $\mathfrak{R}_f$. Thus the results established in the two preceding paragraphs show that the lower bound of H in $\mathfrak{R}_f$ is not less than 1. In particular we have $(Hf, f) \geqq (f, f)$. Since f is an arbitrary element of $\mathfrak{R}$, we must have $\alpha \geqq 1$, as we wished to prove.

We still have to show that, when $\alpha = 1$, this bound is not attained. If it were, Theorem 3 would show that for some $f \neq 0$ the relation $Hf = f$ would hold in $\mathfrak{R}$. Hence we would have $|H^k f| = |f|$ for $k \geqq 1$ in contradiction with the relation $\sigma(f) = +\infty$. The contradiction shows that the bound $\alpha = 1$ is not attained.

5. **Spectral Theory for Bounded Self-Adjoint Operators.** It is now easy to extend Theorem 4 to the general case of a bounded self-adjoint operator A. We have

THEOREM 5. *If A is a self-adjoint operator with finite bounds α and β, there exists a family of projections $E(\lambda)$, $-\infty < \lambda < +\infty$, with the properties*

(1) *$E(\lambda)$ is permutable with A and also with every bounded linear operator which is permutable with A;*
(2) *$\mu \leqq \lambda$ implies $E(\mu)E(\lambda) = E(\mu)$;*
(3) *$E(\lambda) = O$ for $\lambda < \alpha$; $E(\lambda) = I$ for $\lambda \geqq \beta$;*
(4) *if $\epsilon > 0$, then $E(\lambda + \epsilon)f \to E(\lambda)f$ strongly as $\epsilon \to 0$, for every f in $\mathfrak{L}$;*
(5) *in the range of $E(\lambda)$, the upper bound of A does not exceed λ;*
(6) *in the range of $I - E(\lambda)$, the lower bound of A is not less than λ; and, if equal to λ, is not attained.*

The projection $E(\lambda)$ is uniquely determined by properties (1), (5), *and* (6).

Since $H = A - \alpha I$ is a self-adjoint operator with bounds $0, \beta - \alpha$ in accordance with Theorem 1, we can apply to it the results of Theorem 4, constructing the corresponding family of projections $F(\lambda)$. We define $E(\lambda) = F(\lambda - \alpha)$. Properties (2), (3), and (4) are then immediate consequences of the correspond-

ing properties of $F(\lambda)$. Property (1) is a consequence of the permutability of $F(\lambda)$ with $H = A - \alpha I$ and with every bounded linear operator which is permutable with H. Since the upper bound of H in the range of $F(\lambda - \alpha)$ does not exceed $\lambda - \alpha$, the upper bound of $A = H + \alpha I$ in the range of $E(\lambda) = F(\lambda - \alpha)$ does not exceed $\lambda = (\lambda - \alpha) + \alpha$, by virtue of Theorem 1. Thus property (5) is established; and property (6) is proved in similar fashion.

In showing that $E(\lambda)$ is uniquely determined by properties (1), (5), and (6), we may suppose merely that A is a linear symmetric operator. If $E(\lambda)$ and $E'(\lambda)$ are two projections having the indicated properties, then

$$P(\lambda) = E(\lambda)(I - E'(\lambda)) = (I - E'(\lambda))E(\lambda)$$

is a projection with range which is the intersection of the ranges of $E(\lambda)$ and $I - E'(\lambda)$ and which is permutable with A: for the necessary permutability relations follow at once from (1). By (5) and (6), we see that, if the bounds of A in the range of $P(\lambda)$ are defined, the upper bound is not greater than λ and the lower bound not less than λ, being unattained if it is equal to λ. These properties are incompatible, so that the bounds of A are not defined in the range of $P(\lambda)$. Thus the range of $P(\lambda)$ consists of the null element alone, and $P(\lambda) = O$. The relation $E(\lambda) = E(\lambda)E'(\lambda)$ follows immediately. By symmetry we have $E'(\lambda) = E'(\lambda)E(\lambda) = E(\lambda)E'(\lambda) = E(\lambda)$, as we wished to prove.

It is now possible to prove with little difficulty the following approximation theorem:

THEOREM 6. *If A and $E(\lambda)$ have the significance indicated in Theorem 5; if the numbers $\lambda_0 = \alpha, \lambda_1, \cdots, \lambda_{n-1}, \lambda_n = \beta$, arranged in order of magnitude, determine a partition of the interval (α, β), where α and β are the bounds of A; if $\lambda_1', \cdots, \lambda_n'$ are arbitrary numbers such that $\lambda_{k-1} \leqq \lambda_k' \leqq \lambda_k$ for $k = 1, \cdots, n$; if*

$$\delta = \max_{k=1,\cdots,n}(\lambda_k - \lambda_{k-1});$$

and if $E(\Delta_k)$ is defined for the interval $\Delta_k = (\lambda_{k-1}, \lambda_k)$ by the relations

$$E(\Delta_1) = E(\lambda_1), \qquad E(\Delta_k) = E(\lambda_k) - E(\lambda_{k-1}) \text{ for } k = 2, \cdots, n;$$

then $| Af - \sum_{k=1}^{n} \lambda_k' E(\Delta_k) f | \leqq \delta | f |$ for every f in $\mathfrak{L}$.

It is easily seen by properties (1), (2), and (3) of Theorem 5 that the operators $E(\Delta_k)$ are projections permutable with A and satisfying the relations

$$E(\Delta_i)E(\Delta_k) = O \text{ for } i \neq k, \qquad \sum_{k=1}^{n} E(\Delta_k) = I.$$

Thus it is possible by elementary manipulations to write

$$\left| Af - \sum_{k=1}^{n} \lambda_k' E(\Delta_k) f \right|^2 = \sum_{k=1}^{n} | (A - \lambda_k' I) E(\Delta_k) f |^2.$$

The bounds of $A - \lambda_k' I$ in the range of $E(\Delta_k)$ are now seen by virtue of properties (5) and (6) of Theorem 5 and by virtue of Theorem 1 to lie between $-(\lambda_k' - \lambda_{k-1})$ and $\lambda_k - \lambda_k'$: when $k = 2, \cdots, n$, we have only to write

$$E(\Delta_k) = E(\lambda_k)(I - E(\lambda_{k-1}))$$

with the aid of property (2); and when $k = 1$, we use property (5) for the range of $E(\Delta_1) = E(\lambda_1)$ together with the fact that the lower bound of A is $\lambda_0 = \alpha$. Of course, if $E(\Delta_k) = O$ for any k, the corresponding terms in the equality above all vanish and the bounds of A in the range of $E(\Delta_k)$ are not defined. In all cases, we see that Theorem 2 now yields the result $|(A - \lambda_k' I)E(\Delta_k)f| \leqq \delta |E(\Delta_k)f|$. Since we have $\sum_{k=1}^{n} |E(\Delta_k)f|^2 = |f|^2$, we conclude that the inequality stated in the theorem is valid.

An immediate consequence of Theorem 6 is the following result:

THEOREM 7. *If A and $E(\lambda)$ have the significance indicated in Theorem 5, then a bounded linear operator T defined over $\mathfrak{L}$ is permutable with A if and only if it is permutable with $E(\lambda)$ for every λ.*

The necessity of the condition is obvious from Theorem 5, property (1). The sufficiency of the condition follows from the inequality of Theorem 6. If ϵ is any positive number, we determine a partition of the interval (α, β) for which $\delta \leqq \epsilon$ and denote the operator $\sum_{k=1}^{n} \lambda_k' E(\Delta_k)$ of Theorem 6 by A_ϵ. By hypothesis T is permutable with A_ϵ; and there exists a positive number γ such that $|Tg| \leqq \gamma |g|$ for every g in $\mathfrak{L}$. Thus we have

$$|(TA - AT)f| \leqq |T(A - A_\epsilon)f| + |(A - A_\epsilon)Tf|$$
$$\leqq \gamma |(A - A_\epsilon)f| + \delta |Tf| \leqq 2\gamma\delta |f| \leqq 2\gamma\epsilon |f|.$$

Since ϵ is arbitrary we conclude that $(TA - AT)f = 0$ for every f in $\mathfrak{L}$; in other words that $TA = AT$, as we wished to prove.

The inequality of Theorem 6 shows that for a sequence of partitions with δ tending to zero the expression $\sum_{k=1}^{n} \lambda_k' E(\Delta_k)f$ tends strongly to Af. From this fact it is easy to deduce the usual Stieltjes integral representations

$$(Af, g) = \int_\alpha^\beta \lambda d(E(\lambda)f, g), \qquad |Af|^2 = \int_\alpha^\beta \lambda^2 (E(\lambda)f, f).$$

We shall not consider the details, as they are sufficiently familiar from other treatments.

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THE IMBEDDING OF MANIFOLDS IN FAMILIES OF ANALYTIC MANIFOLDS[1]

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I. THE THEOREM

1. **Introduction.** Let $f_1, \cdots, f_{n-m}$ be differentiable functions defined in an open subset of Euclidean n-space E_n. At each point p at which all the f_i vanish, let the gradients $\nabla f_1, \cdots, \nabla f_{n-m}$ be independent. Then the vanishing of the f_i determines a differentiable manifold M of dimension m. Any such manifold we shall say is in "regular position" in E_n. Only certain manifolds are in regular position; see below. The purpose of the paper is to show that any m-manifold M in regular position in E_n may be imbedded in an $(n - m)$-parameter family of homeomorphic analytic manifolds; these fill out a neighborhood of M in E_n.

We may extend the above definition as follows: M is in regular position if, roughly, there exist $n - m$ continuous vector functions in M which, at each point p of M, are independent and independent of vectors determined by pairs of points of M near p. If M is differentiable, the two definitions agree; the ∇f_i are the required vectors. The theorem holds also for this more general class of manifolds.

Some manifolds which are necessarily in regular position if they are differentiable in E_n are:[2]

(a) Any curve; it may have corners.
(b) Any orientable surface.
(c) Any orientable $(n - 1)$-manifold.
(d) Any closed orientable $(n - 2)$-manifold.
(e) Any manifold which is an m-cell.

No non-orientable manifold can be in regular position, and no such manifold can be imbedded in an $(n - m)$-parameter family of manifolds. For examples of orientable manifolds not in regular position, see Sphere-spaces, 8(e) and (f).

Some definitions and lemmas from a paper by the author, Differentiable Manifolds, these Annals, vol. 37 (1936), pp. 645–680, will be used. We refer to this paper as DM. AE will mean the paper referred to in footnote 5 of DM.

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[1] Presented in part to the Am. Math. Soc., Sept. 4, 1934. A statement of the theorem is given in the Proc. Nat. Ac. of Sci., vol. 21 (1935), p. 463.

[2] See H. Whitney, *Sphere-spaces*, Proc. Nat. Ac. of Sci., vol. 21 (1935), p. 467, 8 (d). (e) has been proved by W. Wazewski, Compositio Math., vol. 2 (1935), pp. 63–68. (d) has been proved also by H. Seifert, Math. Zeitschr., vol. 41 (1936), pp. 1–17. He proves here the existence of an analytic (and hence algebraic) approximation for the case of a closed manifold of class C^2.

2. The independence of a set of vectors. This section will be used extensively in part III. With any point p is associated a vector, drawn from the origin to p; with this in mind, we shall use the terms point and vector interchangeably (as long as we do not translate axes). Let $\| v \|$ denote the length of v; then $\| q - p \|$ is the distance from p to q. If v_i is a vector with components $v_{i1}, \cdots, v_{in}$, then from $v_1, \cdots, v_s$ we form the matrix

$$\mathbf{M} = \begin{Vmatrix} v_{11} \cdots v_{1n} \\ \cdots\cdots\cdots \\ v_{s1} \cdots v_{sn} \end{Vmatrix}. \tag{2.1}$$

The square root of the sum of the squares of all s-rowed determinants $D_{i_1 \cdots i_s}$ from $\mathbf{M}$ we call the *independence* of $v_1, \cdots, v_s$; this is also the square root of the determinant of the scalar products of the v's, by the general Lagrange identity:[3]

$$I(v_1, \cdots, v_s) = [\sum D^2_{i_1 \cdots i_s}]^{\frac{1}{2}} = \begin{vmatrix} v_1 \cdot v_1 \cdots v_1 \cdot v_s \\ \cdots\cdots\cdots \\ v_s \cdot v_1 \cdots v_s \cdot v_s \end{vmatrix}^{\frac{1}{2}}. \tag{2.2}$$

$I(v_1, \cdots, v_s)$ is a continuous and symmetric function of $v_1, \cdots, v_s$; it is independent of the (rectangular) coördinate system employed, as each $v_i \cdot v_j$ is. Geometrically, it is the s-dimensional volume of the parallelopiped determined by $v_1, \cdots, v_s$. It vanishes if and only if $v_1, \cdots, v_s$ are dependent.

Some properties of I following at once from (2.2) are:

$$I(v) = \| v \|, \tag{2.3}$$

$$I(v_1 + cv_2, v_2, \cdots) = I(v_1, v_2, \cdots), \tag{2.4}$$

$$I(cv_1, v_2, \cdots) = | c | I(v_1, v_2, \cdots). \tag{2.5}$$

Choose the $x_2, \cdots, x_s$-axes so that $v_2, \cdots, v_s$ lie in the $x_2, \cdots, x_s$-plane, and choose the x_1-axis so that v_1 lies in the $x_1, \cdots, x_s$-plane. Writing out the resulting matrix (2.1) and using the first expression for I in (2.2) gives, as

$$| v_{11} | \leqq \| v_1 \|,$$

$$I(v_1, v_2, \cdots, v_s) \leqq \| v_1 \| I(v_2, \cdots, v_s). \tag{2.6}$$

Repeated use of this inequality gives

$$I(v_1, \cdots, v_s) \leqq \| v_1 \| \cdots \| v_s \|, \tag{2.7}$$

an extension of Hadamard's inequality on determinants. A similar choice of axes with the addition theorem for determinants gives

$$I(v_1' + v_1'', v_2, \cdots) \leqq I(v_1', v_2, \cdots) + I(v_1'', v_2, \cdots). \tag{2.8}$$

Suppose

$$w = \alpha_1 v_1 + \cdots + \alpha_s v_s. \tag{2.9}$$

[3] See O. Schreier and E. Sperner, *Analytische Geometrie* I, Leipzig 1931, pp. 113–114.

By (2.4) and (2.5),

$$I(w, v_2, \cdots, v_s) = I(\alpha_1 v_1, v_2, \cdots, v_s) = |\alpha_1| \, I(v_1, \cdots, v_s).$$

Hence, using (2.6), *if* $I(v_1, \cdots, v_s) \neq 0$,

$$|\alpha_1| \leqq \| w \| \frac{I(v_2, \cdots, v_s)}{I(v_1, \cdots, v_s)}, \tag{2.10}$$

with a similar inequality for each other $|\alpha_i|$. In (2.8), replace v_1' and $v_1' + v_1''$ by v_1 and w_1; we obtain with (2.6)

$$I(w_1, v_2, \cdots) \leqq I(v_1, v_2, \cdots) + \| w_1 - v_1 \| \, I(v_2, \cdots).$$

A similar inequality with v_1 and w_1 interchanged gives, with this,

$$|I(w_1, v_2, \cdots) - I(v_1, v_2, \cdots)| \leqq \| w_1 - v_1 \| \, I(v_2, \cdots). \tag{2.11}$$

An s-fold application gives, using (2.7), *if the* v_i *and* w_i *are unit vectors*,

$$|I(w_1, \cdots, w_s) - I(v_1, \cdots, v_s)| \leqq \| w_1 - v_1 \| + \cdots + \| w_s - v_s \|. \tag{2.12}$$

A vector function $v(p)$ defined in a point set M is of *Class* C^r or *analytic* if each of its components is. It satisfies a *local Lipschitz condition* in M if there exists a positive continuous function $K(p)$ in M such that if p and q are in M, then

$$\| v(q) - v(p) \| \leqq K(p) \| q - p \| \qquad (\| q - p \| < 1/K(p)). \tag{2.13}$$

For this to hold, it is evidently necessary and sufficient that each component of $v(p)$ satisfy a local Lipschitz condition.

3. **Manifolds in regular position in** E_n. Let M be a point set in Euclidean n-space E_n. We shall say M is an *m-manifold in regular position in E_n* if the following is true:

(1) Each point p of M is in a neighborhood U of p in M which is an m-cell.[4]

(2) There are $n - m$ continuous unit vector functions $v_1(p), \cdots, v_{n-m}(p)$ and a positive continuous function $\rho(p) \leqq 1$ defined in M, such that if p and q are in M and $\| q - p \| < \rho(p)$, then

$$I[v_1(p), \cdots, v_{n-m}(p), q - p] \geqq \rho(p) \| q - p \|. \tag{3.1}$$

If there is a limit set LM (see DM §2), we take $\rho(p)$ less than the distance from p to LM.

(3.1) says that for a certain $\epsilon > 0$, the vector $q - p$ makes an angle $\geqq \epsilon$ with the $(n - m)$-plane determined by $v_1(p), \cdots, v_{n-m}(p)$. From (3.1) we find, using (2.6),

$$I[v_1(p), \cdots, v_{n-m}(p)] \geqq \rho(p), \tag{3.2}$$

and thus $v_1(p), \cdots, v_{n-m}(p)$ are linearly independent.

[4] That is, U is homeomorphic with the set of points $\sum_{i=1}^{m} x_i^2 < 1$ in m-space.

If M is of class $C^r(r \geqq 1)$, an equivalent formulation of (2) is evidently: If v is any unit vector parallel to M at p, then

$$I[v_1(p), \cdots, v_{n-m}(p), v] \geqq \rho(p). \tag{3.3}$$

In other words, the tangent plane to M at p and the plane of $v_1(p), \cdots, v_{n-m}(p)$ are independent.

Let M and M^* be manifolds of class $C^r(r \geqq 0)$, and let $\eta(p)$ be a positive continuous function defined in M. We say *M^* approximates to M through the r^{th} order with an error $< \eta(p)$* if M and M^* are homeomorphic and the following is true:

(A) If p and p^* are corresponding points of M and M^*, then $\| p^* - p \| < \eta(p)$.

(B) If $r > 0$, let T be the m-plane tangent to M at p, and let $x_1', \cdots, x_n'$ be a set of rectangular axes such that T is the $x_1', \cdots, x_m'$-plane. Let

$$x_i' = f_i(x_1', \cdots, x_m'), \qquad x_i' = f_i^*(x_1', \cdots, x_m') \qquad (i = m+1, \cdots, n)$$

be functions (of class C^r) determining a neighborhood of p in M and a neighborhood of the corresponding point p^* of M^*. Then if q and q^* are the projections of p and p^* into T,

$$| D_k f_i^*(q^*) - D_k f_i(q) | < \eta(p) \qquad (\sigma_k \leqq r; i = m+1, \cdots, n). \tag{3.4}$$

For the notation, see DM §6, or AE.

THEOREM I. *Let M be an m-manifold of class $C^r(r \geqq 0$ and finite) in regular position in E_n, and let $\eta(p)$ be any positive continuous function defined in M. Then M can be imbedded in an $(n-m)$-parameter family of manifolds*

$$M(c_1, \cdots, c_{n-m}), \qquad |c_i| < 1 \qquad (i = 1, \cdots, n-m) \tag{3.5}$$

with the following properties:

(1) $M(0, \cdots, 0) = M$.

(2) *$M(c_1, \cdots, c_{n-m})$ is analytic if $(c_1, \cdots, c_{n-m}) \neq (0, \cdots, 0)$.*

(3) *Each manifold $M(c_1, \cdots, c_{n-m})$ approximates to M through the r^{th} order with an error $< \eta(p)$.*

(4) *If*

$$p^* = Z(p; c_1, \cdots, c_{n-m}) \tag{3.6}$$

is the point of $M(c_1, \cdots, c_{n-m})$ corresponding to the point p of M, then Z is a continuous function of its arguments, and the points p^* fill out an open set R containing M in a one-one way; hence no two of the manifolds intersect. The points p^* for p fixed fill out a portion of an $(n-m)$-plane through p.

(5) *If the sets $(c_1^{(i)}, \cdots, c_{n-m}^{(i)})(i = 1, 2, \cdots)$ approach the set $(c_1, \cdots, c_{n-m})$, then the manifolds $M(c_1^{(i)}, \cdots, c_{n-m}^{(i)})$ approach the manifold $M(c_1, \cdots, c_{n-m})$, together with partial derivatives of order $\leqq r$.*

(6) *If M is of class C^∞ and has a boundary*[4a] *M', then we can make higher and*

[4a] Boundary shall throughout mean limit set.

higher partial derivatives of $M(c_1, \cdots, c_{n-m})$ *approximate to the corresponding derivatives of* M *as we approach* M'.

II. The Proof, M Differentiable

4. **The vector functions** $w_i(p)$. Set $s = n - m$. The vector functions $v_1(p), \cdots, v_s(p)$ defined in M may not satisfy a local Lipschitz condition; we shall replace them by unit vector functions $w_1(p), \cdots, w_s(p)$ which do, and will in a certain sense be analytic.

Let R_1 be an open set containing M but no boundary points of M. (For instance, R_1 = all points whose distances from M are less than half their distances to the boundary M' of M if M' exists; otherwise, $R_1 = E_n$.) Extend each component $v_{ij}(p)$ of each $v_i(p)$ so it will be continuous in R_1.[5] Set $\zeta(p) = \rho(p)/(4ns)$ and extend it so it will be positive and continuous in R_1. Define the analytic functions $w'_{ij}(p)$ in R_1 by AE Lemma 6 (DM Lemma 9) 1 so that

$$|\, w'_{ij}(p) - v_{ij}(p) \,| < \zeta(p) \text{ in } R_1. \tag{4.1}$$

Now for any p in M,

$$\begin{aligned} |\, ||\, w'_i(p)\,|| - 1 \,| &= |\, ||\, w'_i(p)\,|| - ||\, v_i(p)\,||\,| \leqq ||\, w'_i(p) - v_i(p)\,|| \\ &\leqq \sum_j |\, w'_{ij}(p) - v_{ij}(p)\,| < n\zeta(p) < 1, \end{aligned}$$

and hence $||\, w'_i(p)\,|| \neq 0$ in a neighborhood R_2 of M. Define the unit vector function $w_i(p) = w'_i(p)/||\, w'_i(p)\,||$. Then

$$w_i - v_i = \frac{w'_i}{||\, w'_i\,||}\,(1 - ||\, w'_i\,||) + (w'_i - v_i),$$

$$||\, w_i(p) - v_i(p)\,|| \leqq n\zeta(p) + n\zeta(p) = \rho(p)/(2s) \text{ in } M.$$

Hence, using (2.12), if u is a unit vector,

$$|\, I[w_1(p), \cdots, w_s(p), u] - I[v_1(p), \cdots, v_s(p), u]\,| \leqq \tfrac{1}{2}\rho(p), \tag{4.2}$$

which shows that (3.3) holds with $\rho(p)$ replaced by $\frac{1}{2}\rho(p)$. Therefore M is in regular position in terms of $w_1, \cdots, w_s$. As the first partial derivatives of the w_{ij} are bounded in a neighborhood of any point of R_2, the law of the mean shows that each w_{ij} and hence each w_i satisfies a local Lipschitz condition in R_2 and hence in M: There is a continuous function $K(p) \geqq 1$ in M such that

$$||\, w_i(q) - w_i(p)\,|| \leqq K(p)\,||\, q - p\,|| \qquad (||\, q - p\,|| < 1/K(p)). \tag{4.3}$$

5. **The functions** $\phi_i(p)$. Given any point p of M and any set of numbers $\alpha_1, \cdots, \alpha_s (s = n - m)$, define the point p^* by

$$p^* = p + \sum_{i=1}^{s} \alpha_i w_i(p) = G(p, \alpha_1, \cdots, \alpha_s). \tag{5.1}$$

We wish to solve this equation for p and the α_i in terms of p^*.

[5] Almost any method of extending continuous functions can be used to do this.

LEMMA 1. *There is an open set R_3 containing M, and a vector function H and functions ϕ_i, all of class C^r,*

$$p = H(p^*), \qquad \alpha_i = \phi_i(p^*) \qquad (i = 1, \cdots, s) \tag{5.2}$$

defined in R_3, such that

$$H[p + \sum \alpha_i w_i(p)] = p, \qquad \phi_k[p + \sum \alpha_i w_i(p)] = \alpha_k \tag{5.3}$$

if the point in brackets is in R_3, and conversely, if p^* is in R_3,

$$H(p^*) + \sum \phi_i(p^*) w_i[H(p^*)] = p^*. \tag{5.4}$$

This follows at once from the proof of DM Lemma 21.

LEMMA 2. *Let $T'(p)$ be the s-plane through p in M determined by $w_1(p), \cdots, w_s(p)$, let $x_1', \cdots, x_n'$ be rectangular axes chosen so that $T'(p)$ is the $x_1', \cdots, x_s'$-plane with p at the origin, and with x_1' in the $w_1(p)$-direction, x_2' in the plane of $w_1(p)$ and $w_2(p)$, etc. Then if $\phi_i(x_1, \cdots, x_n) = \phi_i'(x_1', \cdots, x_n'; p)$,*

$$J' = \frac{\partial(\phi_1', \cdots, \phi_s')}{\partial(x_1', \cdots, x_s')} \neq 0. \tag{5.5}$$

If we add $hw_k(p)$ to both sides of (5.1), we obtain

$$p^* + hw_k(p) = p + \sum_{i=1}^{s} (\alpha_i + \delta_{ik} h) w_i(p).$$

This, with (5.3), gives

$$\phi_i[p^* + hw_k(p)] = \phi_i(p^*) + \delta_{ik} h, \qquad \text{if } p = H(p^*). \tag{5.6}$$

From this it follows that

$$\nabla\phi_i \cdot w_k(p) = \delta_{ik}, \tag{5.7}$$

$$\frac{\partial \phi_i'}{\partial x_k'} = 0 \ (k < i) \text{ and } \neq 0 \ (k = i),$$

and hence $J' \neq 0$.

LEMMA 3. *Let $T(p)$ be the tangent plane at p in M, and let $\bar{x}_1, \cdots, \bar{x}_n$ be rectangular axes chosen so that $T(p)$ is the $\bar{x}_1, \cdots, \bar{x}_m$-plane, with p at the origin. Then if $\phi_i(x_1, \cdots, x_n) = \bar{\phi}_i(\bar{x}_1, \cdots, \bar{x}_n; p)$,*

$$\bar{J}|_p = \left.\frac{\partial(\bar{\phi}_1, \cdots, \bar{\phi}_s)}{\partial(\bar{x}_{m+1}, \cdots, \bar{x}_n)}\right|_p \neq 0. \tag{5.8}$$

By the last lemma (using $x_1', \cdots, x_n'$ in place of $x_1, \cdots, x_n$)

$$I(\nabla\phi_1, \cdots, \nabla\phi_s) \neq 0 \text{ on } M. \tag{5.9}$$

Now express I in terms of the $\bar{x}_1, \cdots, \bar{x}_n$ coördinate system. As $\partial\bar{\phi}_i/\partial\bar{x}_j = 0 (j \leqq m)$ at p, all determinants in the new expression except possibly the determinant in (5.8) vanish; hence this determinant in absolute value equals I, and (5.8) is proved.

6. **Two lemmas on implicit functions.** We shall use here the symbols σ_k', D_k' if n is replaced by m.

LEMMA 4. *Let the functions*

$$\Phi_i(x_1, \cdots, x_s) \qquad (i = 1, \cdots, s)$$

be defined and of class C^1 in a neighborhood[5a] *V of the origin q_0, let them vanish at q_0, and let the Jacobian be $\neq 0$ there. Then for any $\epsilon > 0$ there is a $\delta > 0$ with the following property: If $\Theta_1, \cdots, \Theta_s$ are defined and of class C^1 in V, and*

$$|\Theta_i - \Phi_i| < \delta, \qquad \left|\frac{\partial\Theta_i}{\partial x_j} - \frac{\partial\Phi_i}{\partial x_j}\right| < \delta \qquad (i, j = 1, \cdots, s) \tag{6.1}$$

in V, then the Jacobian of the Θ_i is $\neq 0$, and there is a unique point q^* in V such that

$$\| q^* - q_0 \| < \epsilon, \qquad \Theta_i(q^*) = 0 \qquad ((i = 1, \cdots, s). \tag{6.2}$$

Moreover, δ can be made continuous in the Φ_i, their derivatives, and ϵ; the point q^* is continuous in the Θ_i.

LEMMA 5. *Set $s = n - m$. Let the functions*

$$\Phi_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, s)$$

be defined and of class C^r $(r \geqq 1)$ in V, the set of all points within a distance ρ of the origin p_0, let them vanish at p_0, and let

$$J_{p_0} = \left.\frac{\partial(\Phi_1, \cdots, \Phi_s)}{\partial(x_{m+1}, \cdots, x_n)}\right|_{p_0} \neq 0.$$

Then for every $\epsilon > 0$ there is a $\delta > 0$ with the following property: If $\Theta_1, \cdots, \Theta_s$ are defined and of class C^r in V, if

$$|D_k\Theta_i(p) - D_k\Phi_i(p)| < \delta \qquad (\sigma_k \leqq r; i = 1, \cdots, s) \tag{6.3}$$

in V, if $p^* = (x_1^*, \cdots, x_n^*)$ and

$$\| p^* - p_0 \| < \delta, \qquad \Theta_i(p^*) = 0 \qquad (i = 1, \cdots, s), \tag{6.4}$$

and if

$$\psi_i(x_1, \cdots, x_m), \qquad f_i(x_1, \cdots, x_m) \qquad (i = 1, \cdots, s)$$

are continuous functions defined for $x_1^2 + \cdots + x_m^2 < \delta^2$ such that

$$\psi_i(0, \cdots, 0) = 0, \qquad f_i(x_1^*, \cdots, x_m^*) = x_{m+i}^*, \tag{6.5}$$

$$\Phi_i(x_1, \cdots, x_m, \psi_1, \cdots, \psi_s) \equiv 0, \tag{6.6}$$

$$\Theta_i(x_1, \cdots, x_m, f_1, \cdots, f_s) \equiv 0, \tag{6.7}$$

in $x_1, \cdots, x_m$ for $i = 1, \cdots, s$, then the ψ_i and the f_i are of class C^r and

[5a] Which we take sufficiently small.

$$(6.8) \qquad |\, D'_k f_i(x^*_1, \cdots, x^*_m) - D'_k \psi_i(0, \cdots, 0) \,| < \epsilon \qquad (\sigma'_k \leqq r)$$

for $i = 1, \cdots, s$. *Moreover,* δ *is continuous in* ρ, ϵ, *and the* Φ_i *and their derivatives.*

These lemmas are forms of the implicit function theorem, with uniformity and continuity conditions. They may be proved by standard methods.

7. **Proof of** Theorem I, $r \geqq 1$. Given any point p_0 of M, we may choose axes and define the functions $\bar{\phi}_i(\bar{x}_1, \cdots, \bar{x}_n; p_0)$ as in Lemma 3. Moreover, we may evidently choose such axes in a continuous manner for each point p of a whole neighborhood U of p_0 in M; the $\bar{\phi}_i$ will then be continuous in $x_1, \cdots, x_n$ and p for $(x_1, \cdots, x_n)$ near $(0, \cdots, 0)$ and p in U. Lemma 3 shows that the conditions of Lemma 5 are satisfied; the number ρ may obviously be taken continuous in U. Hence, covering M by a set of neighborhoods U, we see that there is a positive continuous function $\Omega_1(p) < \eta(p)$ in M such that if (6.3) through (6.7) hold with Φ_i and δ replaced by $\bar{\phi}_i$ and $\Omega_1(p_0)$, then (6.8) holds with ϵ replaced by $\eta(p_0)$.

Next, define $T'(p_0)$ and axes $x'_1, \cdots, x'_n$ in a continuous manner for any p in a neighborhood U of p_0 as in Lemma 2. For points of $T'(p_0)$, the functions ϕ'_i are then

$$(7.1) \qquad \phi'_i(x'_1, \cdots, x'_s, 0, \cdots, 0; p_0) = \Phi_i(x'_1, \cdots, x'_s; p_0),$$

are defined in a neighborhood of the origin, and are continuous in $x'_1, \cdots, x'_s, p_0$. As $J' \neq 0$, the conditions of Lemma 4 are satisfied. Hence there is a positive continuous function $\Omega_2(p) < \Omega_1(p)$ in M such that if (6.1) holds for axes and functions corresponding to p_0 in M and with δ replaced by $\Omega_2(p_0)$, then (6.2) holds with ϵ replaced by $\Omega_1(p_0)$.

Next, set $\Omega_2(p) = \Omega_2(H(p))$ in R_3. Set

$$(7.2) \qquad \Omega(p) = \Omega_2(p)/(2\,N_r),$$

where N_r is a number such that if F and F' are the expressions for any function of class C^r in terms of any two sets of rectangular axes, and

$$(7.3) \qquad |\, D_k F \,| \leqq a \ (\sigma_k \leqq r), \qquad \text{then } |\, D_k F' \,| \leqq N_r a \ (\sigma_k \leqq r).$$

Let $\Omega'(p)$ be the minimum of $\Omega(p)$ and the distance from p to M. Then $\Omega'(p)$ is continuous in R_3, zero in M, and positive in $R_3 - M$.

By DM Lemma 9 there are functions $\phi''_i(p)$ analytic in $R_3 - M$ and such that

$$(7.4) \qquad |\, D_k \phi''_i(p) - D_k \phi_i(p) \,| \leqq \Omega'(p) \text{ in } R_3 \qquad (\sigma_k \leqq r;\ i = 1, \cdots, s),$$

or at least in $R_3 - M$. If we set $\phi''_i(p) = \phi_i(p)$ in M, then $\phi''_i(p)$ is of class C^r in R_3, and (7.4) holds in M also, by DM Lemma 10. By DM Lemma 25 there is a positive analytic function $\omega(p)$ in R_3 such that

$$(7.5) \qquad |\, D_k \omega(p) \,| < \Omega(p) \text{ in } R_3 \qquad (\sigma_k \leqq r).$$

Take any set of numbers

$$c = (c_1, \cdots, c_s), \qquad |c_i| < 1 \qquad (i = 1, \cdots, s). \tag{7.6}$$

Corresponding to this set we define in R_3

$$\theta_i(p; c) = \phi_i''(p) - c_i\omega(p) \qquad (i = 1, \cdots, s); \tag{7.7}$$

these functions are analytic in $R_3 - M$. (7.4), (7.5) and (7.6) show that

$$|D_k\theta_i(p; c) - D_k\phi_i(p)| \leqq 2\,\Omega(p) \qquad (\sigma_k \leqq r; i = 1, \cdots, s) \tag{7.8}$$

in R_3. The manifold $M(c)$ is defined as the set of points p in R_3 for which

$$\theta_i(p; c) = 0 \qquad (i = 1, \cdots, s). \tag{7.9}$$

Given p in M and $T'(p)$, choose axes $x_1', \cdots, x_n'$ as before, and consider the functions $\Theta_i(x_1', \cdots, x_s')$, the values of the $\theta_i(p; c)$ on $T'(p)$. (7.2), (7.3) and (7.8) show that (6.1) is satisfied with δ replaced by $\Omega_2(p)$; hence there is a unique point

$$p^* = Z(p; c) \tag{7.10}$$

in $T'(p)$ within $\Omega_1(p)$ of p at which (7.8) is satisfied. Lemma 7 shows also that p^* moves continuously as we move p in M. If we let p^* in $M(c)$ correspond to p in M, Lemma 1 shows that $M(c)$ and M are homeomorphic.

Evidently the function Z is continuous in both p and c. No two of the manifolds $M(c)$, $M(c')(c \neq c')$ have a common point, as follows at once from their definition. Given any point (7.10), the corresponding c is given by:

$$c_i = \phi_i''(p^*)/\omega(p^*).$$

As ϕ_i'' and ω are continuous and R_3 is open, to each p' near p^* there corresponds a $c', |c_i'| < 1$. Hence the points (7.10) fill out a neighborhood R of M.

Take p_0 in M, a small neighborhood U of p_0 in M, and a fixed c. For each p in U there is a $p^* = Z(p; c)$; using the axes of Lemma 3, these points p^* are given by functions f_i as in (6.5). As $\Theta_i(p^*) = \theta_i(p^*; c) = 0$ at these points, (6.7) is satisfied. Setting $c = 0$ shows that (6.6) is also. By (7.8) and (7.3), and as $\|Z(p; c) - p\| < \Omega_1(p)$, we may apply Lemma 5; (6.8) now completes the proof of (3) in Theorem I. Finally, the $M(c)$ are analytic for $c \neq 0$, as the $\theta_i(p; c)$ are analytic in $R_3 - M$. (5) is obviously true. If $r = \infty$ and we wish to have (6) satisfied, we use AE Lemma 6 with $m = \infty$. Theorem I for $r \geqq 1$ is now completely proved.

III. The Proof, M not Differentiable

If M is not differentiable, the functions ϕ_i will be continuous, but not differentiable. However, they will have directional derivatives in the directions of the w_i; we may approximate to the ϕ_i by analytic functions having similar directional derivatives. To show this, we first need some preliminary lemmas.

8. **Averages of a function.** Let $S(p, \lambda)$ denote the set of points q within λ of p, let $S'(p, \lambda)$ denote its boundary, let $V(\lambda)$ be the volume of $S(p, \lambda)$, and let

$V'(\lambda)$ be the $(n-1)$-volume of $S'(p, \lambda)$. If $\phi(p)$ is a function continuous in the open set R, we may define the average

$$A(p, \lambda) = \frac{1}{V(\lambda)} \int_{S(p,\lambda)} \phi(q)\, dq. \tag{8.1}$$

Similarly we define $B(p, \lambda)$, the average of ϕ over $S'(p, \lambda)$. A simple computation (noting that $V'(\lambda) = \partial V(\lambda)/\partial\lambda$) shows that $A(p, \lambda)$ has a continuous derivative with respect to λ, given by

$$\frac{\partial}{\partial\lambda} A(p, \lambda) = \frac{n}{\lambda}[B(p, \lambda) - A(p, \lambda)]. \tag{8.2}$$

Also, $A(p, \lambda)$, as a function of p, has continuous first partial derivatives. (In fact, the directional derivative of $A(p, \lambda)$ in the direction of the vector v is given by taking the formula for $B(p, \lambda)$, and putting in the integrand the factor $\cos\theta$, where θ is the angle between $q - p$ and v.)

9. **Approximation to a certain continuous function by an analytic function.** We prove here

LEMMA 6. *Let $\phi(p)$ be a continuous function defined in the open set R, let $\psi_1(p), \cdots, \psi_s(p)$ be continuous in R, and let $w_1(p), \cdots, w_s(p)$ be continuous unit vector functions in R. Let there be a continuous function $\mu(p) \geqq 1$ in R such that if p and q are in R and*

$$q = p + h[w_i(p) + \eta], \qquad \|q - p\| < 1/\mu(p), \qquad |h| < 2/\mu(p), \tag{9.1}$$

then

$$\left|\frac{\phi(q) - \phi(p)}{h} - \psi_i(p)\right| \leqq \mu(p)\,\|\eta\|, \tag{9.2}$$

for $i = 1, \cdots, s$. Then for any positive continuous function $\zeta(p)$ in R there is an analytic function $\phi''(p)$ in R such that

$$|\phi''(p) - \phi(p)| < \zeta(p), \qquad |\nabla\phi''(p)\cdot w_i(p) - \psi_i(p)| < \zeta(p) \tag{9.3}$$

for any p in R and $i = 1, \cdots, s$.[6]

It is sufficient to find a differentiable function $\phi^*(p)$ such that (9.3) holds with $\zeta(p)$ replaced by $\frac{1}{25}\zeta(p)$; we can then at once find the required analytic function by DM Lemma 9. It may be shown that the analytic approximation to $\phi(p)$ given in the proof of AE Lemma 6 has the required property, but this is not very simple.

We show first that $\phi(p)$ satisfies a local Lipschitz condition. Let $L_1(p) \geqq 1$ be a positive continuous function in R such that

$$|\psi_i(p)| < L_1(p) \qquad (i = 1, \cdots, s). \tag{9.4}$$

[6] The lemma evidently holds still if we add $\frac{1}{2}\zeta(p)$ to the right hand side of (9.2).

Take p and q in R, $\| q - p \| < 1/\mu(p)$. Set

$$h = 2 \| q - p \|, \qquad p^* = p + hw_1(p), \qquad \eta = \frac{q - p^*}{h};$$

then

$$q = p^* + h\eta = p + h[w_1(p) + \eta].$$

Also

$$\| \eta \| \leqq \frac{1}{h}[\| q - p \| + \| p^* - p \|] = \frac{3}{2}.$$

Hence, by (9.2), *if* $\| q - p \| < 1/\mu(p)$, *then*

(9.5) $$| \phi(q) - \phi(p) | < 2[L_1(p) + 3\mu(p)/2] \| q - p \| = L_2(p) \| q - p \|.$$

Take p in R and $\lambda < 1/\mu(p)$, and define $A(p, \lambda)$ by (8.1). (9.5) gives

(9.6) $$| A(p, \lambda) - \phi(p) | < \frac{1}{V(\lambda)} \int_{S(p,\lambda)} \lambda L_2(p)\, dq = \lambda L_2(p),$$

with a similar inequality for $| B(p, \lambda) - \phi(p) |$. Hence, by (8.2), *if we keep* $\lambda < 1/\mu(p)$, *then*

(9.7) $$\left| \frac{\partial}{\partial \lambda} A(p, \lambda) \right| < 2nL_2(p).$$

Let $\zeta'(p)$ be a positive continuous function in R so small that

(9.8) $$\zeta'(p) < 1/\mu(p), \qquad \zeta'(p) < \zeta(p)/[8n^2 L_2(p)],$$

and if $\| q - p \| < \zeta'(p)$, then

(9.9) $$\| w_i(q) - w_i(p) \| < \zeta(p)/[16\, \mu(p)],$$
$$| \psi_i(q) - \psi_i(p) | < \zeta(p)/8, \qquad \mu(q) < 2\mu(p),$$

for $i = 1, \cdots, s$, and if $\| q' - p \| < 2\, \zeta'(p)$, then q' is in R. Let $\lambda(p)$ be any positive differentiable function in R such that

(9.10) $$| \lambda(p) |, \left| \frac{\partial \lambda(p)}{\partial x_j} \right| < \zeta'(p) \qquad (j = 1, \cdots, n)$$

(see DM Lemma 25). We define $\phi^*(p)$ by

(9.11) $$\phi^*(p) = A(p, \lambda(p)).$$

To prove (9.3) with $\zeta(p)$ replaced by $\frac{1}{2}\zeta(p)$, take any i and any p in R. Take any h such that $| h | < \zeta'(q)$ for $\| q - p \| < \lambda(p)$, and set $\xi = hw_i(p)$. Take any q, $\| q - p \| \leqq \lambda(p)$, and set $\eta = w_i(p) - w_i(q)$; then q and $q + \xi$ are in R. and

$$q + \xi = q + h[w_i(q) + \eta].$$

Applying (9.2) and (9.9) gives

$$\left| \frac{\phi(q+\xi) - \phi(q)}{h} - \psi_i(p) \right| < \frac{\mu(q)\,\zeta(p)}{16\,\mu(p)} + \frac{\zeta(p)}{8} < \frac{\zeta(p)}{4}.$$

Choose axes $x_1', \cdots, x_n'$ so that $w_i(p)$ lies in the x_1'-direction. Knowing that $A(p, \lambda)$ is differentiable, the last inequality gives (keeping λ fixed)

$$\left| \frac{\partial}{\partial x_1'} A(p, \lambda) - \psi_i(p) \right| \leqq \frac{\zeta(p)}{4}. \tag{9.12}$$

Hence

$$\left| \frac{\partial \phi^*(p)}{\partial x_1'} - \psi_i(p) \right| < 2n\, L_2(p)\, \frac{n\,\zeta(p)}{8n^2\, L_2(p)} + \frac{\zeta(p)}{4} = \frac{\zeta(p)}{2},$$

proving the second inequality of (9.3). The first follows from (9.6), (9.10) and (9.8). The lemma is now proved.

10. **The functions $\phi_i(p)$.** Given p in M, let $\rho'(p)$ be the smaller of the two numbers $\rho(p)$, $1/K(p)$ (see (4.3)), and take $\rho''(p) \leqq \frac{1}{2}\,\rho'(p)$ positive and continuous in M so that if $\| q - p \| < \rho''(p)$, then

$$\tfrac{1}{2}\,\rho'(p) < \rho'(q) < 2\rho'(p), \qquad \tfrac{1}{2}\,\rho'(p) < \rho(q) < 2\rho(p). \tag{10.1}$$

Let $T^*(p)$ be the set of points (5.1), with

$$|\,\alpha_i\,| < \rho''(p)/[3\,s\,K(p)]. \tag{10.2}$$

LEMMA 7. *Set $K'(p) = 12/\rho(p)$ in M. If p^* and q^* are in $T^*(p)$ and $T^*(q)$ respectively and $\| q^* - p^* \| < \frac{1}{3}\,\rho''(p)$, then*

$$\| q - p \| \leqq K'(p)\, \| q^* - p^* \|. \tag{10.3}$$

Say $q^* = q + \Sigma\, \beta_i w_i(q)$. Subtracting (5.1) from this and using (10.2) gives

$$\| q - p \| < \frac{\rho''(p)}{3} + \frac{s\rho''(p)}{3sK(p)} + \frac{s\rho''(q)}{3sK(q)}.$$

We show now that

$$\| q - p \| < \min\,[\rho'(p), \rho'(q)]. \tag{10.4}$$

According as $\rho''(q) \leqq \rho''(p)$ or $> \rho''(p)$, we find

$$\| q - p \| < \rho''(p) \leqq \tfrac{1}{2}\,\rho'(p) < \rho'(q),$$
$$\| q - p \| < \rho''(q) \leqq \tfrac{1}{2}\,\rho'(q) < \rho'(p).$$

Set

$$\Delta_i = w_i(q) - w_i(p); \tag{10.5}$$

then

$$q - p = q^* - p^* + \sum\,(\alpha_i - \beta_i) w_i(q) - \sum\,\alpha_i \Delta_i.$$

Applying (2.4), (2.7), (4.3) and (10.2) gives

$$\begin{aligned} I &= I[w_1(q), \cdots, w_s(q), q - p] \\ &= I[w_1(q), \cdots, w_s(q), q^* - p^* - \sum \alpha_i \Delta_i] \\ &\leqq \| q^* - p^* \| + \frac{\rho(p)}{6} \| q - p \|. \end{aligned}$$

Also, by (3.1) with v_i and ρ replaced by w_i and $\frac{1}{2}\rho$, and (10.1),

$$I \geqq \| q - p \| \tfrac{1}{2} \rho(q) > \| q - p \| \tfrac{1}{4} \rho(p).$$

That we may apply (4.3) and (3.1) follows from (10.4). The last two inequalities give (10.3).

LEMMA 8. *The sets $T^*(p)$ fill out a neighborhood R' of M in a* 1-1 *way.*

By the last lemma, no two sets $T^*(p)$, $T^*(q)$ $(q \neq p)$ have a common point. To show that R' is an open set, take p_0 in M, and let U be a neighborhood of p_0 in M which is an m-cell. Map U into a portion of E_n with its last $n - m$ coördinates zero. If p in U is mapped into p' in E_n, take any p^* of (5.1) and map it into the point with the same first m coördinates as p' and with $\alpha_2, \cdots, \alpha_s$ as its last coördinates. The resulting region S' of E_n is homeomorphic with the set S of points on $T^*(p)$, p in U. By the theorem on invariance of regions, S is a region. R' is the sum of such sets S, and hence is open.

We may now define the functions ϕ_i and H as in Lemma 1, and they will be single valued and continuous in R'.

11. **The difference quotients of the $\phi_i(p)$.** We first extend the definition of the $w_k(p)$ over R' by setting $w_k(p) = w_k(H(p))$.

LEMMA 9. *The functions $\phi(p) = \phi_k(p)$, $\psi_i(p) = \delta_{ik}$, $w_i(p)(i = 1, \cdots, s)$ satisfy the conditions of Lemma* 6 *for each $k = 1, \cdots, s$ and an open subset R of R' containing M.*

Let R be the set formed by keeping α_i less than one half the right hand side of (10.2). Let $\mu(p)$ be a positive continuous function in M so large that

$$\mu(p) > 9/\rho''(p), \qquad 3[K'(p)\, K(p) + K'(p) + 1]/\rho(p), \tag{11.1}$$

and if $\mu(p^*) = \mu(H(p^*))$ in R and $\| p' - p^* \| < 2/\mu(p^*)$, p^* in R, then p' is in R'. Now take any p^* and q^* in R and any h so that

$$\| q^* - p^* \| < 1/\mu(p^*), \qquad | h | < 2/\mu(p^*). \tag{11.2}$$

Take a fixed i, set $p = H(p^*)$, $q = H(q^*)$, and define $\alpha_j, \beta_j, \eta, p'$ by

$$\begin{aligned} p^* &= p + \sum \alpha_j w_j(p), \qquad q^* = q + \sum (\alpha_j + \beta_j) w_j(q), \\ q^* &= p^* + h[w_i(p) + \eta], \qquad p' = p^* + h w_i(p). \end{aligned} \tag{11.3}$$

As $\| p' - p^* \| < 2/\mu(p^*)$, p' is in R'. By (11.3) and (11.1),

(11.4) $$\| q^* - p' \| \leqq \| q^* - p^* \| + | h | < 3/\mu(p^*) < \tfrac{1}{3}\rho''(p);$$

hence we may apply (10.3), giving, with (11.1),

(11.5) $$\| q - p \| < K'(p) \| q^* - p' \| < 3\, K'(p)/\mu(p) < 1/K(p).$$

This shows that we may apply (4.3), giving with (10.5) and (11.5),

(11.6) $$\| \Delta_j \| \leqq K(p)\, K'(p) \| q^* - p' \| = K(p)\, K'(p) \, | h | \, \| \eta \|,$$

as $q^* - p' = h\eta$. From (11.3) we find

$$q - p = hw_i(p) + h\eta - \sum (\alpha_j + \beta_j)\Delta_j - \sum \beta_j w_j(p).$$

Hence, by (11.5) and (11.6),

$$\| \sum (\beta_j - \delta_{ij}h)w_j(p) \| < [K'(p) + 1 + K(p)\, K'(p)] \, | h | \, \| \eta \|.$$

Dividing by $| h |$ and applying (2.10), (2.7), and (3.2) with v_j and ρ replaced by w_j and $\frac{1}{2}\rho$ gives, with (11.1),

(11.7) $$\| \beta_k/h - \delta_{ik} \| < \mu(p) \| \eta \| \qquad (k = 1, \cdots, s).$$

As $\beta_k = \phi_k(q^*) - \phi_k(p^*)$ and $\mu(p) = \mu(p^*)$, (9.2) is proved.

12. **Proof of** Theorem I, $r = 0$. The proof follows the lines of §7, except that we do not use Lemmas 3 and 5. Given a point p_0 of M, we may obviously take a neighborhood U of p_0 and choose axes $x_1', \cdots, x_n'$ as in Lemma 2 in a continuous manner for p in U. For each p in U we have functions (7.1) satisfying the conditions of Lemma 4, (by Lemma 2, which holds even if $r = 0$). Take $\Omega_2(p) < \eta(p)$ so that if (6.1) holds with δ replaced by $\Omega_2(p)$, then (6.2) holds with ϵ replaced by $\eta(p)$. Set $\Omega(p) = \Omega_2(p)/(2n)$, and define $\Omega'(p)$ as before.

By lemmas 9 and 6 we may define the functions $\phi_i''(x_1, \cdots, x_n)$ in $R_3 - M$ so that, setting $\phi_i'' = \phi_i = 0$ in M,

(12.1) $$| \phi_i''(p) - \phi_i(p) |, \qquad | \nabla\phi_i''(p) \cdot w_k(p) - \delta_{ik} | \leqq \Omega'(p)$$

in R_3; the ϕ_i'' are continuous in R_3. We now define $\omega(p)$ and the $\theta_i(p; c)$ by (7.5) and (7.7). (7.8) is replaced by

(12.2) $$| \theta_i(p; c) - \phi_i(p) |, \qquad | \nabla\theta_i(p; c) \cdot w_k(p) - \delta_{ik} | \leqq 2\,\Omega(p)$$

in R_3. $M(c)$ is defined by (7.9). The rest of the proof goes as before.

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ON THE PROJECTIVE GEOMETRY OF PATHS

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Introduction. The present paper arose from the attempt to understand, purely from the standpoint of the geometry of paths, (that is, without going back to the "tangent spaces" of the underlying manifold) the projective theories of T. Y. Thomas[1] and O. Veblen,[2] and especially the introduction of the additional coördinate x^0 in these theories. Naturally it is only the particular case of a symmetric projective connection with an uniquely determined system of paths[3] which had to be considered. In that attempt I was led to a new conception of a projective geometry of paths which is a generalization of the usual notion. This new conception is explained in §1. Besides the invariance under projective changes of the affine connection, a particular parameter on the paths, called the *projective normal parameter*, here plays a principal part.[4] In §2 I give a first mathematical formulation of the fundamental idea. The projective normal parameter is defined by means of a Schwarzian differential equation up to linear fractional transformations. It is this Schwarzian equation which is fundamental for the following work. Then I show (§3) that Veblen's *gauge variable*[5] is connected with the transition from the affine parameter on the paths to the projective normal parameter, and I consider the place of Veblen's *transformations of the representation*[6] in the projective geometry of paths. In §4 the generalization of the Weyl projective curvature tensor[7] for an arbitrary projective geom-

[1] T. Y. Thomas, *On the projective and equi-projective geometries of paths.* Proc. N. A. S. 11 (1925), pp. 199–203; *A projective theory of affinely connected manifolds.* Math. Zeitschr. 25 (1926), pp. 723–33. See also T. Y. Thomas, *The differential invariants of generalized spaces.* Cambridge University Press 1934, X + 241 pp. This book will be referred to as D. I.

[2] O. Veblen, *Generalized projective geometry.* Journal London Math. Soc. 4 (1929), pp. 140–60; *Projektive Relativitätstheorie.* J. Springer, Berlin, 1933, V + 73 pp. This book will be referred to as P. R.

[3] O. Veblen, P. R., pp. 29–30.

[4] This parameter was introduced by J. H. C. Whitehead, *The representation of projective spaces.* Ann. of Math. 32 (1931), pp. 327–60; see especially §3. An analogous parameter on the geodesics of a general projectively connected manifold was used by Enea Bortolotti, *Sulle connessioni proiettive.* Rend. Circ. Mat. Palermo 56 (1932), pp. 1–57, especially pp. 30–31. For a two-dimensional Finsler space the *preferred* projective normal parameter (see §9) was introduced by Funk under the name of "linearisierender Parameter." See P. Funk, *Über zweidimensionale Finslersche Räume, insbesondere über solche mit geradlinigen Extremalen und positiver konstanter Krümmung.* Math. Zeitschr. 40 (1935), pp. 86–93. Here the Schwarzian differential equation appears which is fundamental for my view of the subject.

[5] P. R., p. 10.

[6] P. R., p. 10.

[7] H. Weyl, *Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung.* Göttinger Nachr. 1921, pp. 99–112, especially p. 101.

etry of paths is given. §5 is devoted to an important particular case in which the projective normal parameter is already determined by the differential equations of the paths alone. In §6 a second method of treating a projective geometry of paths is given which corresponds to the theory of T. Y. Thomas. This method leads very naturally to the generalization of the Weyl curvature tensor, found in §4 (§7). In §8 I discuss briefly the equivalence problem for two projective geometries of paths.

Paragraphs 2–8 deal with the *restricted* geometry of paths in the sense of J. Douglas.[8] In §9 I explain, as concisely as possible, the corresponding ideas for the *general* geometry of paths.

As far as possible, the notations are choosen in conformity with the existing literature, at the risk of being not always quite consistent. When I refer to a function without further qualification, I shall mean an analytic function regular at all points (or line-elements[9]) under consideration. Repeated indices indicate summation.

1. **Projective geometry of paths.** The conception of a projective geometry of paths which we shall develop in the present paper rests on two notions: the notion of *the class of affine connections belonging to a system of paths* and the notion of *a projective normal parameter of a system of paths. The class of affine connections belonging to a system of paths* consists of all symmetric affine connections for which the system of paths is identical with the system of autoparallel curves. Any connection of the class is derived from any other connection of the class by a *projective change of the affine connection.*[10]

A projective normal parameter of a system of paths is a parameter which has the following properties:

(a) It is determined, up to an arbitrary linear fractional transformation, on all paths of the system at the same time.

(b) It is not altered by transformations of coördinates.

(c) It remains the same for all affine connections of the class belonging to the system of paths.

By every one of the ∞^3 possible choices of a projective normal parameter, a *projective scale* is fixed on all paths of the system at the same time.

By a *projective geometry of paths* we understand *the simultaneous theory of the class of affine connections belonging to a system of paths and of a projective normal parameter of this system.*

In the preceding statements, we can understand the notion of a system of paths in the restricted or in the general sense.[11] Corresponding to this, we must understand the notion of the affine connection in the restricted or in the general sense. An affine connection is called *restricted* if its components are point functions, *general* if they are functions of the line-element.[12]

[8] J. Douglas, *The general geometry of paths.* Ann. of Math. (2) 29 (1928), pp. 143–68.

[9] In §9.

[10] H. Weyl, l.c., pp. 99–100.

[11] Cf. J. Douglas, l.c., pp. 144.

[12] See §9.

2. **Restricted projective geometry of paths.** We consider an n-dimensional manifold $\mathfrak{M}$ referred to a definite coördinate system (x^i). By a *path* (in the *restricted* sense[13]) we denote a curve $x^i = x^i(t)$ given as solution of a system of differential equations

$$\frac{dx^i}{dt}\left(\frac{d^2x^l}{dt^2} + \Gamma^l_{mr}\frac{dx^m}{dt}\frac{dx^r}{dt}\right) - \frac{dx^l}{dt}\left(\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\right) = 0. \tag{2.1}$$

Here the Γ^i_{jk} are given point functions symmetric in the lower indices:

$$\Gamma^i_{jk} = \Gamma^i_{kj}. \tag{2.2}$$

On introducing a new parameter s by means of the substitution

$$\frac{\dfrac{d^2s}{dt^2}}{\dfrac{ds}{dt}} = \frac{\dfrac{d^2x^i}{dt^2} + \Gamma^i_{jk}\dfrac{dx^j}{dt}\dfrac{dx^k}{dt}}{\dfrac{dx^i}{dt}}, \tag{2.3}$$

we can write the differential equations (2.1) in the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{2.4}$$

s is called an *affine* parameter of the system of paths. It is determined up to entire linear transformations.

The equations (2.1) remain unaltered if we put, in the place of the functions Γ^i_{jk}, the new functions

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} - \psi_j\,\delta^i_k - \psi_k\,\delta^i_j \tag{2.5}$$

where the ψ_i are arbitrary point functions and the δ^i_k Kronecker symbols. To the functions $\hat{\Gamma}^i_{jk}$ corresponds a new affine parameter $\hat{s}$. If we introduce $\hat{s}$ as independent variable, the differential equations of the paths take the form

$$\frac{d^2x^i}{d\hat{s}^2} + \hat{\Gamma}^i_{jk}\frac{dx^j}{d\hat{s}}\frac{dx^k}{d\hat{s}} = 0. \tag{2.6}$$

On the other hand, the equations (2.4) change[14] to

$$\frac{d^2x^i}{d\hat{s}^2} + \frac{\hat{s}''}{(\hat{s}')^2}\frac{dx^i}{d\hat{s}} + \Gamma^i_{jk}\frac{dx^j}{d\hat{s}}\frac{dx^k}{d\hat{s}} = 0, \tag{2.7}$$

by the introduction of $\hat{s} = \hat{s}(s)$, and these equations must be identical with (2.6). With regard to (2.5), this gives

$$\frac{d\hat{s}}{ds} = e^{-2\int\psi_j dx^j}. \tag{2.8}$$

When we transform the coördinates x^i by any analytical transformation

$$\bar{x}^i = \bar{x}^i\,(x^1, x^2, \cdots x^n) \tag{2.9}$$

[13] J. Douglas, l.c., p. 144.

[14] The dashes signify differential coefficients: $\hat{s}' = d\hat{s}/ds$, $s'' = d^2\hat{s}/ds^2$.

whose functional determinant is different from zero for all points under consideration, the Γ^i_{jk} transform according to the equations

$$\bar{\Gamma}^l_{jk} = \left(\Gamma^i_{mr}\frac{\partial x^m}{\partial \bar{x}^j}\frac{\partial x^r}{\partial \bar{x}^k} + \frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k}\right)\frac{\partial \bar{x}^l}{\partial x^i}. \tag{2.10}$$

They are, therefore, the components of a (symmetric) affine connection Γ. The same is true for the $\hat{\Gamma}^i_{jk}$. Consequently, the ψ_i in (2.5) are the components of a covariant vector. The transition from the Γ^i_{jk} to the $\hat{\Gamma}^i_{jk}$ according to (2.5) is called a *projective change of the affine connection* Γ.[15] *The class K of affine connections belonging to the system* (2.1) *of paths* (§1) consists of all affine connections given by (2.1), (2.2) and (2.5).

Now we define a *projective normal parameter* π on the paths (2.1) by means of a differential equation of the form

$$\{\pi, s\} = -2M\,\Gamma^0_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}. \tag{2.11}$$

In (2.11), $\{v, u\}$ is the Schwarzian derivative

$$\{v, u\} = \frac{\dfrac{d^3v}{du^3}}{\dfrac{dv}{du}} - \frac{3}{2}\left(\frac{\dfrac{d^2v}{du^2}}{\dfrac{dv}{du}}\right)^2,$$

M is a constant not zero[16] and the Γ^0_{jk} are given point functions symmetric in the lower indices:

$$\Gamma^0_{jk} = \Gamma^0_{kj}. \tag{2.12}$$

By (2.11) π is defined as function of s up to linear fractional transformations.

We demand that the parameter π remain unaltered by transformations of coördinates and by arbitrary projective changes of the affine connection Γ (or, what amounts to the same, by the introduction of another affine parameter $\hat{s}$ on the paths). From the first of this postulates, it follows that the Γ^0_{jk} are transformed by transformations of coördinates according to

$$\bar{\Gamma}^0_{jk} = \Gamma^0_{lm}\frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^m}{\partial \bar{x}^k}, \tag{2.13}$$

because the affine parameter s remains unaltered (up to entire linear transformations) by the transformations (2.9) of coördinates. The Γ^0_{jk} are, therefore, the components of a covariant tensor of rank 2. Moreover, by putting

$$\{\pi, \hat{s}\} = -2M\,\hat{\Gamma}^0_{jk}\frac{dx^j}{d\hat{s}}\frac{dx^k}{d\hat{s}}, \qquad (\hat{\Gamma}^0_{jk} = \hat{\Gamma}^0_{kj}) \tag{2.14}$$

---

[15] H. Weyl, l.c., pp. 99–100.

[16] Without loss of generality, we could put $M = 1$. We do not particularise M with regard to later considerations (§6).

and by taking notice of

$$\{\pi, \hat{s}\} = \left(\frac{ds}{d\hat{s}}\right)^2 [\{\pi, s\} - \{\hat{s}, s\}], \tag{2.15}$$

we find from the second postulate by means of (2.8),

$$\hat{\Gamma}^0_{jk} = \Gamma^0_{jk} - \frac{1}{M}\left\{\frac{1}{2}\left(\frac{\partial \psi_j}{\partial x^k} + \frac{\partial \psi_k}{\partial x^j}\right) - \psi_r \Gamma^r_{jk} + \psi_j \psi_k\right\}. \tag{2.16}$$

(2.16) are the equations of transformation of the Γ^0_{jk} by a projective change (2.5) of the affine connection Γ.

Hence, *a projective geometry of paths in* $\mathfrak{M}$ (*in the restricted sense*) *according to the definition of* §1 *can be fixed by giving* $\frac{1}{2}n(n+1)^2$ *point functions* $\Gamma^i_{jk}(x)$, $\Gamma^0_{jk}(x)$, *symmetric in the lower indices and satisfying the laws of transformations* (2.10), (2.5), (2.13), (2.16).

If we perform successively, in an arbitrary order, a projective change (2.5) of the affine connection and a transformation (2.9) of coördinates the Γ^i_{jk}, Γ^0_{jk} change respectively to

$$\hat{\bar{\Gamma}}^i_{jk} = \left(\Gamma^l_{mr}\frac{\partial x^m}{\partial \bar{x}^j}\frac{\partial x^r}{\partial \bar{x}^k} + \frac{\partial^2 x^l}{\partial \bar{x}^j \partial \bar{x}^k}\right)\frac{\partial \bar{x}^i}{\partial x^l} - \psi_l\left(\frac{\partial x^l}{\partial \bar{x}^j}\delta^i_k + \frac{\partial x^l}{\partial \bar{x}^k}\delta^i_j\right), \tag{2.17}$$

$$\hat{\bar{\Gamma}}^0_{jk} = \left\{\Gamma^0_{lm} - \frac{1}{M}\left[\frac{1}{2}\left(\frac{\partial \psi_l}{\partial x^m} + \frac{\partial \psi_m}{\partial x^l}\right) - \psi_r \Gamma^r_{lm} + \psi_l \psi_m\right]\right\}\frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^m}{\partial \bar{x}^k}. \tag{2.18}$$

Here, the $\hat{\bar{\Gamma}}^i_{jk}$, $\hat{\bar{\Gamma}}^0_{jk}$ are functions of the $\bar{x}^i$, whilst the Γ^i_{jk}, Γ^0_{jk}, ψ_l are functions of the x^i.

The method explained above is not the only one for treating analytically a projective geometry of paths. Another method is given in §6.

3. **The gauge variable and the transformations of the representation.** We start with a definitely choosen projective normal parameter π. Corresponding to the transition from the projective normal parameter π to the affine parameter s, we introduce on every path a *gauge variable* x^0 by the substitution

$$x^0 = -\frac{1}{2M}\log\frac{ds}{d\pi}.^{17} \tag{3.1}$$

Let $\hat{s}$ be another affine parameter on the paths and

$$\bar{x}^0 = -\frac{1}{2M}\log\frac{d\hat{s}}{d\pi} \tag{3.2}$$

---

[17] A different definition of the gauge variable has been given by Hlavatý, by means of a density of weight -1, respectively $-1/M(n+1)$ (in our notation). See V. Hlavatý, *Connexion projective et dêplacement projectif*. Annali di Mat. (4) 12 (1933–4), pp. 217–94, especially pp. 224–5; *Système complet des invariants d'une courbe dans un espace projectif courbe*. Abh. a. d. Seminar für Vektor- und Tensoranalysis II.–III., Moskau und Leningrad 1935, pp. 119–150, especially p. 121.

the gauge variable corresponding to the transition $\pi \to \bar{s}$. Then it follows from (2.8) that for any path

$$\bar{x}^0 = x^0 + \frac{1}{M}\int_q^x \psi_i(x)\,dx^i + C, \tag{3.3}$$

where the integral is taken from an arbitrary but fixed point q on the path along the path and where C signifies an arbitrary constant. Generally $\psi_i(x)dx^i$ is not an exact differential. For $\psi_i(x) = 0$, the equation (3.3) contains also the special case that $\bar{s}$ is nothing but another determination $as + b$ (a, b constants, $a \neq 0$) of the *same* affine parameter; in this case $C = -(\log a)/2M$.

To the different determinations of the same affine parameter therefore corresponds the continuous group $\bar{x}^0 = x^0 + t$.

Let us now introduce the parameter π as independent variable in the differential equations (2.4) and (2.11). Since

$$\{v, u\} = -\left(\frac{dv}{du}\right)^2 \{u, v\},$$

we have

$$\{s, \pi\} = 2M\Gamma^0_{jk}\frac{dx^j}{d\pi}\frac{dx^k}{d\pi}. \tag{3.4}$$

On replacing in (3.4) and (2.4) the derivations $ds/d\pi$, $d^2s/d\pi^2$, $d^3s/d\pi^3$ by x^0, $dx^0/d\pi$, $d^2x^0/d\pi^2$ according to (3.1), we find that the equations (3.4) and (2.4) can be written respectively as

$$\frac{d^2x^0}{d\pi^2} + M\left(\frac{dx^0}{d\pi}\right)^2 + \Gamma^0_{jk}\frac{dx^j}{d\pi}\frac{dx^k}{d\pi} = 0, \tag{3.5}$$

$$\frac{d^2x^i}{d\pi^2} + 2M\frac{dx^0}{d\pi}\frac{dx^i}{d\pi} + \Gamma^i_{jk}\frac{dx^j}{d\pi}\frac{dx^k}{d\pi} = 0. \tag{3.6}$$

Finally, we adopt the convention that Latin indices will take on values from $1, \cdots, n$ and Greek indices values from $0, 1, \cdots, n$, and define the functions $\Pi^\alpha_{\beta\gamma}$ by

$$\text{(a)}\quad \Pi^\alpha_{jk} = \Gamma^\alpha_{jk}, \tag{3.7}$$

$$\text{(b)}\quad \Pi^\alpha_{\beta 0} = \Pi^\alpha_{0\beta} = M\delta^\alpha_\beta.$$

Then the equations (3.6) take the form

$$\frac{d^2x^\alpha}{d\pi^2} + \Pi^\alpha_{\beta\gamma}\frac{dx^\beta}{d\pi}\frac{dx^\gamma}{d\pi} = 0. \tag{3.8}$$

Obviously it is also possible to make the steps which led from (2.4) and (2.11) to (3.8) in the inverse order.

Hence, *a projective geometry of the paths* (2.1) *can be regarded as the theory of the*

differential equations (3.8), *with the additional conditions* $\partial\Pi^{\alpha}_{\beta\gamma}/\partial x^0 = 0$ *and* (3.7b), *under the transformations*

$$\begin{cases} d\bar{x}^0 = dx^0 + \dfrac{1}{M}\psi_i(x)\,dx^i, \\ \bar{x}^i = \bar{x}^i(x^1, x^2, \cdots, x^n), & \left(\dfrac{\partial(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)}{\partial(x^1, x^2, \cdots, x^n)} \neq 0\right) \end{cases} \tag{3.9}$$

where the $\psi_i(x)$ *are arbitrary functions of* $x^1, x^2, \cdots, x^n$.

If the x^{α} are interpreted as coördinates in an $(n+1)$-dimensional manifold ${}^*\mathfrak{M}$ the equations (3.9) arise then and only then from a transformation of coördinates in ${}^*\mathfrak{M}$ when $\psi_i(x)dx^i$ is the exact differential of a function of $x^1, x^2, \cdots x^n$. For

$$\psi_i = M\,\frac{\partial \log \rho}{\partial x^i}, \qquad (\rho = \rho(x^1, x^2, \cdots, x^n)) \tag{3.10}$$

we have from (3.9)

$$\begin{cases} \bar{x}^0 = x^0 + \log \rho(x^1, x^2, \cdots, x^n), \\ \bar{x}^i = \bar{x}^i(x^1, x^2, \cdots, x^n), & \left(\dfrac{\partial(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)}{\partial(x^1, x^2, \cdots, x^n)} \neq 0\right). \end{cases} \tag{3.11}$$

Here $\rho(x^1, x^2, \cdots, x^n)$ is an arbitrary function of its arguments.

The transformations (3.11) are Veblen's *transformations of the representation*. Thus, *to restrict one's-self to the transformations of the representation means, that one does not consider the total class* K *of all affine connections belonging to the system* (2.1) *of paths, but restricts one's-self to the subclass* K_1 *of the affine connections arising from an arbitrary but fixed affine connection* Γ *of the class* K *by the following special projective changes of the affine connection*:

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} - M\,\frac{\partial \log \rho}{\partial x^j}\,\delta^i_k - M\,\frac{\partial \log \rho}{\partial x^k}\,\delta^i_j, \tag{3.12}$$

(ρ arbitrary function of $x^1, x^2, \cdots, x^n$).

Otherwise expressed: We obtain the transformations of the representation by considering the subclass K_1 as representative for the whole class K.

It is easy to characterize the subclass K_1. Let

$$B^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^i_{ml}\Gamma^m_{jk} - \Gamma^i_{mk}\Gamma^m_{jl} \tag{3.13}$$

be the curvature tensor of the affine connection Γ. Then the subclass K_1 consists of all affine connections of the class K which possess the same contracted curvature tensor

$$B^i_{ijk} = B_{kj} - B_{jk}, \qquad (B_{kj} = B^i_{kji}) \tag{3.14}$$

as the affine connection Γ.[18]

[18] See L. P. Eisenhart, *Non-Riemannian Geometry*, New York, American Mathematical Society, 1927, VIII + 184 pp., especially p. 91. This book will be referred to as N. R. G.

The $\Pi^{\alpha}_{\beta\gamma}$ are transformed by the transformations of the representation according to

$$\bar{\Pi}^{\alpha}_{\beta\gamma} = \left(\Pi^{\mu}_{\sigma\tau}\frac{\partial x^{\sigma}}{\partial \bar{x}^{\beta}}\frac{\partial x^{\tau}}{\partial \bar{x}^{\gamma}} + \frac{\partial^2 x^{\mu}}{\partial \bar{x}^{\beta}\,\partial \bar{x}^{\gamma}}\right)\frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}}. \tag{3.15}$$

4. **The curvature tensor of a projective geometry of paths.** We shall now derive a tensor in $\mathfrak{M}$ which is invariant under arbitrary projective changes of the affine connection Γ and represents the natural generalization of the Weyl projective curvature tensor.

For that purpose, we consider the $\Pi^{\alpha}_{\beta\gamma}$, defined by (3.7). According to (3.15), the $\Pi^{\alpha}_{\beta\gamma}$ are transformed, by the transformations of the representation (3.11), like the components of an affine connection Π in the $(n+1)$-dimensional manifold $^*\mathfrak{M}$ (§3). The components $R^{\alpha}_{\beta\gamma\delta}$ of the curvature tensor of the connection Π vanish identically save R^{0}_{jkl} and[19]

$$R^{i}_{jkl} = B^{i}_{jkl} + M\Gamma^{0}_{jk}\delta^{i}_{l} - M\Gamma^{0}_{jl}\delta^{i}_{k}. \tag{4.1}$$

In (4.1), B^{i}_{jkl} is the curvature tensor (3.13) of the affine connection Γ in $\mathfrak{M}$. The R^{i}_{jkl} are the components of a tensor in $\mathfrak{M}$. By a projective change (2.5) of the affine connection Γ the tensor R^{i}_{jkl} transforms into

$$\hat{R}^{i}_{jkl} = R^{i}_{jkl} + \tfrac{1}{2}\psi_{jk}\delta^{i}_{l} - \tfrac{1}{2}\psi_{jl}\delta^{i}_{k} - \psi_{kl}\delta^{i}_{j}, \tag{4.2}$$

where

$$\psi_{jk} = \frac{\partial \psi_{j}}{\partial x^{k}} - \frac{\partial \psi_{k}}{\partial x^{j}}. \tag{4.3}$$

Equation (4.2) can be derived with the aid of (4.1), (2.16) and [20]

$$\hat{B}^{i}_{jkl} = B^{i}_{jkl} + \left(\frac{\partial \psi_{j}}{\partial x^{k}} - \psi_{r}\Gamma^{r}_{jk} + \psi_{j}\psi_{k}\right)\delta^{i}_{l} - \left(\frac{\partial \psi_{j}}{\partial x^{l}} - \psi_{r}\Gamma^{r}_{jl} + \psi_{j}\psi_{l}\right)\delta^{i}_{k} - \psi_{kl}\delta^{i}_{j}. \tag{4.4}$$

From (4.2), it follows that the contracted tensor

$$R_{jk} = R^{i}_{jki} = B_{jk} + M(n-1)\Gamma^{0}_{jk} \tag{4.5}$$

is transformed into

$$\hat{R}_{jk} = R_{jk} + \frac{n+1}{2}\psi_{jk}, \tag{4.6}$$

by a projective change of the affine connection Γ. By introducing the value $\psi_{jk} = \frac{1}{2}(\psi_{jk} - \psi_{kj})$ from (4.6) in (4.2), we see that the tensor

$$\begin{aligned} {}^*B^{i}_{jkl} = R^{i}_{jkl} &- \frac{1}{n+1}\frac{R_{jk} - R_{kj}}{2}\delta^{i}_{l} + \frac{1}{n+1}\frac{R_{jl} - R_{lj}}{2}\delta^{i}_{k} \\ &+ \frac{1}{n+1}(R_{kl} - R_{lk})\,\delta^{i}_{j} \end{aligned} \tag{4.7}$$

[19] See O. Veblen, l.c., Journ. London Math. Soc., p. 157; P. R., p. 20.

[20] See L. P. Eisenhart, N. R. G., p. 87. Eisenhart's notations differ slightly from ours.

remains unaltered by arbitrary projective changes of the affine connection Γ. If we replace the R^i_{jkl} and R_{jk} in (4.7) by their values (4.1), (4.5) we get finally

$$
\begin{aligned}
{}^*B^i_{jkl} = B^i_{jkl} &- \left(\frac{1}{n+1}\frac{B_{jk}-B_{kj}}{2} - M\Gamma^0_{jk}\right)\delta^i_l \\
&+ \left(\frac{1}{n+1}\frac{B_{jl}-B_{lj}}{2} - M\Gamma^0_{jl}\right)\delta^i_k + \frac{1}{n+1}(B_{kl}-B_{lk})\,\delta^i_j.
\end{aligned} \tag{4.8}
$$

We call the tensor with the components ${}^*B^i_{jkl}$ the *curvature tensor of the projective geometry of paths* under consideration. It has the following properties

$$
\begin{aligned}
&\text{(a)} \quad {}^*B^i_{ikl} = 0, \\
&\text{(b)} \quad {}^*B^i_{jkl} + {}^*B^i_{jlk} = 0, \\
&\text{(c)} \quad {}^*B^i_{jkl} + {}^*B^i_{klj} + {}^*B^i_{ljk} = 0.
\end{aligned} \tag{4.9}
$$

In §7 we come back to it once more.

5. **The preferred projective normal parameter.** From (4.9, a, c), we see that the contracted curvature tensor ${}^*B_{jk} = {}^*B^i_{jki}$ of a projective geometry of paths is symmetric. From (4.7) and (4.8), it follows that

$$
{}^*B_{jk} = \tfrac{1}{2}(R_{jk} + R_{kj}) = \tfrac{1}{2}(B_{jk} + B_{kj}) + M\,(n-1)\,\Gamma^0_{jk}. \tag{5.1}
$$

By the equation

$$
{}^*B_{jk} = 0 \tag{5.2}
$$

a *preferred projective normal parameter* is defined (up to linear fractional transformations) which is already fully determined by the class K of affine connections belonging to the system (2.1) of paths. From (5.1), (5.2), we have

$$
\Gamma^0_{jk} = -\frac{1}{M(n-1)}\frac{B_{jk}+B_{kj}}{2}, \tag{5.3}
$$

and by substitution of this value in (2.11) we get the differential equation

$$
\{\pi, s\} = \frac{2}{n-1}B_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} \tag{5.4}
$$

which defines the *preferred projective normal parameter.*

The class K of affine connections belonging to a system (2.1) of paths together with the preferred projective parameter determines a *special* projective geometry of paths. The curvature tensor of this special projective geometry of paths is the Weyl projective curvature tensor:[21]

$$
\begin{aligned}
W^i_{jkl} = B^i_{jkl} &+ \frac{1}{n^2-1}[(nB_{jl}+B_{lj})\,\delta^i_k - (nB_{jk}+B_{kj})\,\delta^i_l] \\
&+ \frac{1}{n+1}(B_{kl}-B_{lk})\,\delta^i_j.
\end{aligned} \tag{5.5}
$$

[21] H. Weyl, l.c., p. 101.

Indeed the equation (4.8) changes into (5.5), by introduction of the value (5.3) for Γ^0_{jk}. Equation (5.2) gives $W_{jk} = W^i_{jki} = 0$.

The curvature tensor ${}^*B^i_{jkl}$ of an *arbitrary* projective geometry of paths and the Weyl projective curvature tensor are connected by

$$W^i_{jkl} = {}^*B^i_{jkl} - \frac{{}^*B_{jk}}{n-1}\delta^i_l + \frac{{}^*B_{jl}}{n-1}\delta^i_k, \tag{5.6}$$

as follows from (4.8), (5.1), (5.5). Thus, the equation

$${}^*B^i_{jkl} = 0 \tag{5.7}$$

is equivalent to the two equations

$$W^i_{jkl} = 0, \qquad {}^*B_{jk} = 0. \tag{5.8}$$

6. **The method of T. Y. Thomas.** A second way of dealing with a projective geometry of paths results from a slight generalization of the method used by T. Y. Thomas in the research of projectively connected manifolds.[22]

Let us call every system of functions $\hat{\Gamma}^i_{jk}(x)$, resulting in a fixed coördinate system (x^i) from (2.5) by a definite choice of the functions ψ_i, a *connection of the class K in the coördinate system* (x^i), independently of the manner in which the ψ_i and consequently the $\hat{\Gamma}^i_{jk}$ are transformed by transformations of coördinates. Under these connections there exists a single *normal connection* characterized[23] by $\hat{\Gamma}^i_{ij} = 0$. We denote this connection by Π and its components by Π^i_{jk},[24] so that we have

$$\Pi^i_{ij} = 0. \tag{6.1}$$

A simple calculation gives

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{1}{n+1}\Gamma^m_{mj}\delta^i_k - \frac{1}{n+1}\Gamma^m_{mk}\delta^i_j. \tag{6.2}$$

Consequently Π corresponds to the choice

$$\psi_i = \frac{1}{n+1}\Gamma^m_{mi} \tag{6.3}$$

of the ψ_i in (2.5). The corresponding parameter $\hat{s}$ is the *projective parameter* p of T. Y. Thomas. According to (2.8) and (6.3) we have

$$\frac{dp}{ds} = e^{-\frac{2}{n+1}\int \Gamma^m_{mj}dx^j}. \tag{6.4}$$

---

[22] T. Y. Thomas, l.c. See particularly the exposition of this method in D. I., Chapter I.–III., and the discussion of J. H. C. Whitehead, l.c., §10.

[23] See E. Cartan, *Sur les variétés à connexion projective.* Bull. Soc. Math. France 52 (1924), pp. 205–41; J. M. Thomas, *Asymmetric displacement of a vector.* Trans. Amer. Math. Soc. 28 (1926), pp. 658–70, especially p. 664. Cf. also L. P. Eisenhart, N. R. G., p. 104.

[24] The Π^i_{jk} of this and the following paragraphs have nothing to do with the Π^i_{jk} of §3.

With the parameter p as independant variable the differential equations (2.1) of the paths take the form

$$\frac{d^2x^i}{dp^2} + \Pi^i_{jk}\frac{dx^j}{dp}\frac{dx^k}{dp} = 0. \tag{6.5}$$

The *projective normal parameter* π can now be defined by the differential equation

$$\{\pi, p\} = -2M\,\Pi^0_{jk}\frac{dx^j}{dp}\frac{dx^k}{dp}, \tag{6.6}$$

up to linear fractional transformations. In (6.6), the Π^0_{jk} are point functions symmetric in the lower indices:

$$\Pi^0_{jk} = \Pi^0_{kj}. \tag{6.7}$$

The connection between the Π^0_{jk} and the Γ^i_{jk}, Γ^0_{jk} (§2) is given by (2.16) if we substitute the Π^0_{jk} for the $\hat{\Gamma}^0_{jk}$ and the values (6.3) for the ψ_i. Thus, we get

$$\Pi^0_{jk} = \Gamma^0_{jk} - \frac{1}{M(n+1)}\left\{\frac{1}{2}\left(\frac{\partial\Gamma^m_{mj}}{\partial x^k} + \frac{\partial\Gamma^m_{mk}}{\partial x^j}\right) - \Gamma^l_{lm}\,\Gamma^m_{jk} + \frac{1}{n+1}\,\Gamma^l_{lj}\,\Gamma^m_{mk}\right\}. \tag{6.8}$$

By this method, the class K of the affine connections belonging to the system (2.1) of paths is represented in each coördinate system by a single connection, the normal connection Π. But, the condition $\hat{\Gamma}^i_{ij} = 0$ being invariant only under transformations of coördinates with constant jacobian and not under arbitrary transformations of coördinates, the Π^i_{jk} are *not* transformed by such transformations like the components of an affine connection, the projective parameter p is *not* invariant under transformations of coördinates and the functions Π^0_{jk} are *not* the components of a covariant tensor of rank 2. The formulas of transformation for the Π^i_{jk}, Π^0_{jk}, p by transformations of coördinates are rather

$$\bar{\Pi}^l_{jk} = \left(\Pi^h_{mr}\frac{\partial x^m}{\partial\bar{x}^j}\frac{\partial x^r}{\partial\bar{x}^k} + \frac{\partial^2 x^h}{\partial\bar{x}^j\,\partial\bar{x}^k}\right)\frac{\partial\bar{x}^l}{\partial x^h} - \frac{1}{n+1}\frac{\partial\log(x\bar{x})}{\partial x^j}\delta^l_k - \frac{1}{n+1}\frac{\partial\log(x\bar{x})}{\partial\bar{x}^k}\delta^l_j, \tag{6.9}$$

$$\bar{\Pi}^0_{jk} = \Pi^0_{lm}\frac{\partial x^l}{\partial\bar{x}^j}\frac{\partial x^m}{\partial\bar{x}^k} + \frac{1}{M(n+1)}\left\{\left(\Pi^h_{lm}\frac{\partial x^l}{\partial\bar{x}^j}\frac{\partial x^m}{\partial\bar{x}^k} + \frac{\partial^2 x^h}{\partial\bar{x}^j\,\partial\bar{x}^k}\right)\frac{\partial\bar{x}^r}{\partial x^h}\frac{\partial\log(x\bar{x})}{\partial\bar{x}^r} - \frac{1}{n+1}\frac{\partial\log(x\bar{x})}{\partial\bar{x}^j}\frac{\partial\log(x\bar{x})}{\partial\bar{x}^k} - \frac{\partial^2\log(x\bar{x})}{\partial\bar{x}^j\,\partial\bar{x}^k}\right\}, \tag{6.10}$$

$$\frac{d\bar{p}}{dp} = C\,(x\bar{x})^{-\frac{2}{n+1}}, \qquad (C = \text{const.}). \tag{6.11}$$

Here and in the following developments, $(\bar{x}x)$, $(x\bar{x})$ are respectively the functional determinants of the transformation (2.9) of coördinates and of the inverse transformation.

Here, the transition to the variable x^0 takes place by means of the equation

$$x^0 = -\frac{1}{2M}\log\frac{dp}{d\pi}, \tag{6.12}$$

corresponding to (3.1). The variable x^0 corresponding to another determination $ap + b$(a, b constants, $a \neq 0$) of the projective parameter p is $x^0 - (\log a)/2M$. By repeating the calculus of §3 with p, Π^0_{jk}, Π^i_{jk} instead of s, Γ^0_{jk}, Γ^i_{jk} respectively and by introducing the convention of §3 on the Latin and Greek indices, we can put the system of the differential equations (6.5), (6.6) in the form

$$\frac{d^2x^\alpha}{d\pi^2} + {}^*\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\pi}\frac{dx^\gamma}{d\pi} = 0. \tag{6.13}$$

Here the ${}^*\Gamma^\alpha_{\beta\gamma}$ are defined by

$$\begin{cases} {}^*\Gamma^\alpha_{jk} = \Pi^\alpha_{jk}, \\ {}^*\Gamma^\alpha_{\beta 0} = {}^*\Gamma^\alpha_{0\beta} = M\,\delta^\alpha_\beta. \end{cases} \tag{6.14}$$

Since the parameter p is not invariant under transformations of coördinates with variable jacobian, the variable x^0 corresponding to the transition $\pi \to p$ underlies not only the transformations $\bar{x}^0 = x^0 + \text{const.}$, but the transformations

$$\bar{x}^0 = x^0 - \frac{1}{M(n+1)}\log(\bar{x}x) + \text{const.}, \tag{6.15}$$

which follow from (6.11) and (6.12). Corresponding to this, the transformations (3.9) are here replaced by the transformations

$$\begin{cases} \bar{x}^0 = x^0 - \dfrac{1}{M(n+1)}\log(\bar{x}x) + \text{const.}, \\ \bar{x}^i = \bar{x}^i(x^1, x^2, \cdots, x^n). \qquad ((\bar{x}x) \neq 0). \end{cases} \tag{6.16}$$

From (6.13) and from the invariance of the parameter π under transformations of coördinates, it follows that the ${}^*\Gamma^\alpha_{\beta\gamma}$ are transformed by the transformations (6.16) according to

$${}^*\bar{\Gamma}^\alpha_{\beta\gamma} = \left({}^*\Gamma^\mu_{\sigma\tau}\frac{\partial x^\sigma}{\partial\bar{x}^\beta}\frac{\partial x^\tau}{\partial\bar{x}^\gamma} + \frac{\partial^2 x^\mu}{\partial\bar{x}^\beta\partial\bar{x}^\gamma}\right)\frac{\partial\bar{x}^\alpha}{\partial x^\mu}. \tag{6.17}$$

The theory of T. Y. Thomas results if we take $M = -1/(n+1)$ and determine the functions Π^0_{jk} from the connection Π by the equation

$$\Pi^0_{jk} = \frac{n+1}{n-1}\mathfrak{B}_{jk} \tag{6.18}$$

corresponding to (5.3). Thereby, $\mathfrak{B}^i_{jkl}$ is the *equiprojective curvature tensor*, that is the curvature tensor of the connection Π, and

$$\mathfrak{B}_{jk} = \mathfrak{B}^i_{jki} = \frac{\partial \Pi^i_{jk}}{\partial x^i} - \Pi^i_{mk}\Pi^m_{ji}, \tag{6.19}$$

so that

$$\mathfrak{B}_{jk} = \mathfrak{B}_{kj}. \tag{6.20}$$

7. **The curvature tensor of the connection** $^*\Gamma$. The equation (6.17) expresses that the functions $^*\Gamma^\alpha_{\beta\gamma}$ defined by (6.14) behave, under the transformations (6.16), like the components of an affine connection $^*\Gamma$ in the $(n+1)$-dimensional manifold $^*\mathfrak{M}$ with coördinates x^α. The components $^*B^\alpha_{\beta\gamma\delta}$ of the curvature tensor of this connection $^*\Gamma$ vanish identically, with the exception of $^*B^0_{jkl}$, $^*B^i_{jkl}$. These components are

$$^*B^i_{jkl} = \mathfrak{B}^i_{jkl} + M\Pi^0_{jk}\delta^i_l - M\Pi^0_{jl}\delta^i_k, \tag{7.1}$$

$$^*B^0_{jkl} = \frac{\partial \Pi^0_{jk}}{\partial x^l} - \frac{\partial \Pi^0_{jl}}{\partial x^k} + \Pi^m_{jk}\Pi^0_{ml} - \Pi^m_{jl}\Pi^0_{mk}. \tag{7.2}$$

From (6.20), we see that the equation (7.1) results if we substitute in (4.8), for the Γ^0_{jk}, Γ^i_{jk}, the Π^0_{jk}, Π^i_{jk} (§6) respectively. The $^*B^i_{jkl}$ are, therefore, the components of the *curvature tensor of the projective geometry of paths* under consideration, introduced in §4.[25]

The $^*B^0_{jkl}$ transform by the transformations (6.16) according to

$$^*\bar{B}^0_{jkl} = \left(^*B^0_{qrt} + {}^*B^i_{qrt}\frac{\partial \bar{x}^0}{\partial x^i}\right)\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^t}{\partial \bar{x}^l}. \tag{7.3}$$

Hence, they are then and only then the components of a tensor in $\mathfrak{M}$ when

$$^*B^i_{jkl} = 0. \tag{7.4}$$

For $n > 2$ we have

$$M{}^*B^0_{jkl} = \frac{1}{n+1}\Gamma^i_{im}{}^*B^m_{jkl} - \frac{1}{n-2}({}^*B^i_{jkl,i} - {}^*B_{jk,l} + {}^*B_{jl,k}), \tag{7.5}$$

where the comma denotes the covariant derivative with respect to the Γ^i_{jk}. In this case, we see that (7.4) has the consequence $^*B^0_{jkl} = 0$. For $n = 2$, we get

$$\begin{aligned} M{}^*B^0_{jkl} = {} & \tfrac{1}{3}(\Gamma^i_{il}{}^*B_{jk} - \Gamma^i_{ik}{}^*B_{jl}) + {}^*B_{jk,l} - {}^*B_{jl,k} \\ & - \tfrac{2}{3}(B_{jk,l} - B_{jl,k}) - \tfrac{1}{3}(B_{kj,l} - B_{lj,k}). \end{aligned} \tag{7.6}$$

[25] This method goes back to T. Y. Thomas, l.c., Math. Zeitschr., p. 732. Cf. also O. Veblen, *Projective tensors and connections*. Proc. N. A. S. 14 (1928), pp. 154–166, especially §8.—(7.2) corresponds to the expression given by J. M. Thomas, for the Weyl projective curvature tensor; see J. M. Thomas, *Note on the projective geometry of paths*. Proc. N. A. S. 11 (1925), pp. 207–9.

(7.6) follows, for $n = 2$, from the general formula

$$M^*B^0_{jkl} = \frac{1}{n+1}\Gamma^i_{im}{}^*B^m_{jkl} + \frac{1}{n^2-1}\{(n+1)(^*B_{jk,l} - {}^*B_{jl,k}) - n(B_{jk,l} - B_{jl,k}) - (B_{kj,l} - B_{lj,k})\}, \tag{7.7}$$

from which also (7.5) can be derived by means of

$$^*B^i_{jkl,i} = \frac{n-2}{n^2-1}\{(B_{kj,l} - B_{lj,k}) + n(B_{jk,l} - B_{jl,k})\} + \frac{1}{n-1}(^*B_{jk,l} - {}^*B_{jl,k}).^{26} \tag{7.8}$$

From known theorems on the projectively flat spaces[27] and from the remark at the end of §5, it follows finally, with the aid of (7.5) and (7.6):

For $n > 2$, the vanishing of the curvature tensor of a projective geometry of paths is a necessary and sufficient condition that the class K of affine connections belonging to the system of paths contain an euclidean-affine connection and that the affine parameter on the paths corresponding to this connection be at the same time a projective normal parameter. For $n = 2$, the condition (7.4) must be replaced by

$$^*B^i_{jkl} = 0, \qquad ^*B^0_{jkl} = 0. \tag{7.9}$$

8. **The equivalence problem.** We call two projective geometries of paths in an n-dimensional manifold $\mathfrak{M}$ *equivalent* if they have the same paths and the same projective normal parameter on the paths (up to linear fractional transformations). The problem: to decide if two given projective geometries of paths are equivalent or not, is the *equivalence problem.* It is solved if we can give necessary and sufficient conditions for the equivalence of two projective geometries of paths. Since this equivalence problem is not essentially different from the equivalence problem for two projectively connected manifolds,[28] we can be brief.

Let two projective geometries of paths be given by functions $\Gamma^i_{jk}(x)$, $\Gamma^0_{jk}(x)$ and $\bar{\Gamma}^i_{jk}(\bar{x})$, $\bar{\Gamma}^0_{jk}(\bar{x})$ respectively, in the manner of §2. Let $^*\Gamma^\alpha_{\beta\gamma}(x)$ be the functions defined by (6.14), (6.2), (6.8) and $^*\bar{\Gamma}^\alpha_{\beta\gamma}(x)$ the functions formed in the same manner by means of the $\bar{\Gamma}^i_{jk}(\bar{x})$, $\bar{\Gamma}^0_{jk}(\bar{x})$. Then we find without difficulty from §6, (2.17), (2.18): *A necessary and sufficient condition for the equivalence of the two projective geometries of paths is that there exist, in the $(n+1)$-dimensional manifold $^*\mathfrak{M}$ with coördinates x^α (§6), a transformation of the form (6.16) such that the $^*\Gamma^\alpha_{\beta\gamma}(x)$ and the $^*\bar{\Gamma}^\alpha_{\beta\gamma}(\bar{x})$ are connected by (6.17).*

[26] For the special case $M = -1/(n+1)$, $\Pi^0_{jk} = [(n+1)/(n-1)]\,\mathfrak{B}_{jk}$ see T. Y. Thomas, D. I., p. 58, formulas (18.5)–(18.7).

[27] H. Weyl, l.c., pp. 104–5.

[28] O. Veblen and J. M. Thomas, *Projective invariants of affine geometry of paths.* Ann. of Math. (2) 27 (1926), pp. 279–96; T. Y. Thomas, l.c., Math. Zeitschr.; J. H. C. Whitehead, l.c. Cf. also T. Y. Thomas, D. I., §§16–19, 88.

Now J. H. C. Whitehead proved the following theorem:[29] *Let, in the* $(n+1)$*-dimensional manifold* $^*\mathfrak{M}$*, the connections* $^*\Gamma$ *and* $^*\bar{\Gamma}$*, with components* $^*\Gamma^\alpha_{\beta\gamma}$ *and* $^*\bar{\Gamma}^\alpha_{\beta\gamma}$ *respectively, be affinely equivalent, that is, let there exist in* $^*\mathfrak{M}$ *a transformation of coördinates*

$$\bar{x}^\alpha = \bar{x}^\alpha(x^0, x^1, \cdots, x^n) \tag{8.1}$$

with not-vanishing jacobian such that the $^*\Gamma^\alpha_{\beta\gamma}$ *and* $^*\bar{\Gamma}^\alpha_{\beta\gamma}$ *are connected by* (6.17). *Then there always exists at least one transformation of the particular form* (6.16) *possessing the same property.*

With the aid of this theorem we have finally:

A necessary and sufficient condition for the equivalence of two projective geometries of paths in an n*-dimensional manifold* $\mathfrak{M}$*, given by* $\Gamma^i_{jk}(x)$, $\Gamma^0_{jk}(x)$ and $\bar{\Gamma}^i_{jk}(\bar{x})$, $\bar{\Gamma}^0_{jk}(\bar{x})$ *respectively, is that, in the* $(n+1)$*-dimensional manifold* $^*\mathfrak{M}$ *with coordinates* x^α*, the connections* $^*\Gamma$, $^*\bar{\Gamma}$ *defined by* (6.2), (6.8), (6.14) *and the corresponding equations be affinely equivalent.*

By this theorem the equivalence problem for two projective geometries of paths in an n-dimensional manifold $\mathfrak{M}$ is reduced to the affine equivalence problem for two particular affine connections in an $(n+1)$-dimensional manifold.[30]

9. **General projective geometry of paths.** A *general affine connection* in an n-dimensional manifold $\mathfrak{M}$ is a geometric object Γ whose components $\Gamma^i_{jk}(x, \dot{x})$ in a determined coördinate system (x^i) are functions of the $2n$ variables x^i, $\dot{x}^i$, homogeneous of the degree zero in the $\dot{x}^i$, and are transformed by the transformations (2.9) of coördinates according to (2.10) if the $\dot{x}^i$ are transformed like the components of a contravariant vector. We consider only symmetric general affine connections, that is, such that (2.2) is valid.

By a *path* in the *general* sense we understand an integral curve $x^i = x^i(t)$ of a system of differential equations of the form

$$\frac{dx^i}{dt}\left(\frac{d^2x^l}{dt^2} + 2\Gamma^l\left(x, \frac{dx}{dt}\right)\right) - \frac{dx^l}{dt}\left(\frac{d^2x^i}{dt^2} + 2\Gamma^i\left(x, \frac{dx}{dt}\right)\right) = 0, \tag{9.1}$$

where the functions $\Gamma^i(x, \dot{x})$ are homogeneous of the second degree in the $\dot{x}^i$. By the substitution

$$\frac{\dfrac{d^2s}{dt^2}}{\dfrac{ds}{dt}} = \frac{\dfrac{d^2x^i}{dt^2} + 2\Gamma^i\left(x, \dfrac{dx}{dt}\right)}{\dfrac{dx^i}{dt}}, \tag{9.2}$$

the equations (9.1) take the form

$$\frac{d^2x^i}{ds^2} + 2\Gamma^i\left(x, \frac{dx}{ds}\right) = 0. \tag{9.3}$$

[29] J. H. C. Whitehead, l.c., §8.

[30] For this problem see e.g. T. Y. Thomas, D. I. §§81–85.

s is called an *affine* parameter of the system (9.1) of paths. It is determined up to entire linear transformations.

Every affine connection Γ such that

$$\Gamma^i_{jk}(x, \dot{x})\dot{x}^j\dot{x}^k = 2\Gamma^i(x, \dot{x}) \tag{9.4}$$

has not only the paths (9.1) for autoparallel curves but corresponds moreover to the same affine parameter s.[31] Under these connections Γ, the connection with the components $\Gamma^i_{jk} = \partial^2\Gamma^i/\partial\dot{x}^i\partial\dot{x}^k$ is preferred.

The equations (9.1) do not change if we put, in the place of the functions Γ^i, the new functions

$$\hat{\Gamma}^i(x, \dot{x}) = \Gamma^i(x, \dot{x}) - \psi(x, \dot{x})\dot{x}^i, \tag{9.5}$$

where $\psi(x, \dot{x})$ is an arbitrary function homogeneous of the first degree in the $\dot{x}^i$. To the functions $\hat{\Gamma}^i$ corresponds the form

$$\frac{d^2x^i}{d\hat{s}^2} + 2\hat{\Gamma}^i\left(x, \frac{dx}{d\hat{s}}\right) = 0 \tag{9.6}$$

of the differential equation of the paths, with the new affine parameter $\hat{s}$. Along any path, the connection between s and $\hat{s}$ is given by

$$\frac{d\hat{s}}{ds} = e^{-2\int\psi(x,dx)}. \tag{9.7}$$

The class K of affine connections belonging to the system (9.1) *of paths consists of all symmetric affine connections $\hat{\Gamma}$ whose components $\hat{\Gamma}^i_{jk}(x, \dot{x})$ satisfy the conditions*

$$\hat{\Gamma}^i_{jk}\dot{x}^j\dot{x}^k = \Gamma^i_{jk}\dot{x}^j\dot{x}^k - 2\psi\dot{x}^i. \tag{9.8}$$

In (9.8), the Γ^i_{jk} are the components of an arbitrary symmetric general affine connection Γ satisfying (9.4) and ψ has the meaning stated above.[32] With transformations of coördinates, ψ is a scalar.

A projective normal parameter π on the paths (9.1) can be defined by

$$\{\pi, s\} = -4M\Gamma^0\left(x, \frac{dx}{ds}\right), \tag{9.9}$$

up to linear fractional transformations. In (9.9) M is a given constant different from zero and $\Gamma^0(x, \dot{x})$ a given function homogeneous of the second degree in the

[31] An example of two general symmetric affine connections related in this way is given by the two connections with the components G^k_{ij} and Γ^{*k}_{ij} in E. Cartan, *Les espaces de Finsler*, Actualités scientifiques et industrielles 79, Hermann et Cie, Paris 1934, 42 pp., especially pp. 18–9.

[32] For a general geometry of paths, it would be preferable to take as the starting point of the theory not the components Γ^i_{jk} of the connections satisfying (9.4) but the functions Γ^i themselves, considered as the components of a geometric object in the coördinate system (x^i).

$\dot{x}^i$. We ask that the parameter π shall not alter by transformations of coordinates and by projective changes of the affine connection Γ according to (9.8) (or, what amounts to the same, by transition to another affine parameter $\hat{s}$ on the paths). This gives for Γ^0 the law of transformation

$$\bar{\Gamma}^0(\bar{x}, \dot{\bar{x}}) = \Gamma^0(x, \dot{x}), \qquad \left(\dot{\bar{x}}^i = \frac{\partial \bar{x}^i}{\partial x^k}\, \dot{x}^k\right) \tag{9.10}$$

by the transformations (2.9) of coördinates and

$$\hat{\Gamma}^0 = \Gamma^0 - \frac{1}{2M}\left(\frac{\partial \psi}{\partial x^i}\, \dot{x}^i - 2\, \frac{\partial \psi}{\partial \dot{x}^i}\, \Gamma^i + \psi^2\right) \tag{9.11}$$

by a projective change of the affine connection Γ according to (9.8). (9.11) follows by means of (2.15). In the functions on both sides of the equation (9.11) the arguments are x^i, $\dot{x}^i$.

We now introduce the gauge variable x^0 by means of (3.1), take the parameter π as independent variable in (9.3) and (9.9) and adopt the convention of §3 on the Latin and Greek indices. Then we can comprehend the equations (9.3) and (9.9) in the form

$$\frac{d^2x^\alpha}{d\pi^2} + 2P^\alpha\left(x, \frac{dx}{d\pi}\right) = 0, \tag{9.12}$$

where the $P^\alpha(x, \dot{x})$ are defined by

$$P^0(x, \dot{x}) = \tfrac{1}{2}M(\dot{x}^0)^2 + \Gamma^0(x, \dot{x}), \tag{9.13}$$

$$P^i(x, \dot{x}) = M\dot{x}^0\dot{x}^i + \Gamma^i(x, \dot{x}). \tag{9.14}$$

In an analogous manner as in §3, we find here:

A projective geometry of the paths (9.1) *can be considered as the theory of the differential equations* (9.12) *where the* P^α *are given by* (9.13), (9.14), *under the transformations*

$$\begin{cases} d\bar{x}^0 = dx^0 + \dfrac{1}{M}\,\psi(x, dx), \\ \bar{x}^i = \bar{x}^i(x^1, x^2, \cdots, x^n), \end{cases} \qquad ((\bar{x}x) \neq 0). \tag{9.15}$$

In (9.15), $\psi(x, dx)$ *is an arbitrary function of the* x^i, dx^i, *homogeneous of the first degree in the* dx^i.

The equations (9.15) arise then and only then from a transformation of coordinates in the $(n + 1)$-dimensional manifold ${}^*\mathfrak{M}$ (§3) when we restrict ourselves to the changes

$$\hat{\Gamma}^i = \Gamma^i - M\, \frac{\partial \log \rho}{\partial x^j}\, \dot{x}^j\dot{x}^i \tag{9.16}$$

of the functions Γ^i, where ρ is an arbitrary function of $x^1, x^2, \cdots, x^n$ alone. This signifies that we consider only the subclass K_1 of the class K of affine connections belonging to the system (9.1) of paths which consists of all affine connec-

tions of the class K connected with an arbitrary but definitely choosen affine connection Γ of this class by

$$\hat{\Gamma}^i_{jk}\dot{x}^j\dot{x}^k = \Gamma^i_{jk}\dot{x}^j\dot{x}^k - 2M\frac{\partial \log \rho}{\partial x^j}\dot{x}^j\dot{x}^i. \tag{9.17}$$

Then the transformations (9.15) reduce to the transformations of the representation (3.11). By a transformation of the representation, the P^α transform according to

$$\begin{cases} \bar{P}^\alpha(\bar{x},\dot{\bar{x}}) = \left(P^\tau(x,\dot{x}) + \frac{1}{2}\frac{\partial^2 x^\tau}{\partial\bar{x}^\beta\,\partial\bar{x}^\gamma}\dot{\bar{x}}^\beta\dot{\bar{x}}^\gamma\right)\frac{\partial\bar{x}^\alpha}{\partial x^\tau}, \\ \left(\dot{\bar{x}}^\lambda = \frac{\partial\bar{x}^\lambda}{\partial x^\mu}\dot{x}^\mu\right). \end{cases} \tag{9.18}$$

The $\dfrac{\partial^2 P^\tau}{\partial\dot{x}^\alpha\,\partial\dot{x}^\beta}$ transform, consequently, like the components of a general affine connection in the manifold $^*\mathfrak{M}$.

In the general projective geometry of paths, there exists also a *preferred projective normal parameter* which is the generalization of that introduced in §5. For it can be shown without difficulty that the scalar[33]

$$2B = 2\frac{\partial\Gamma^i}{\partial x^i} - \frac{\partial^2\Gamma^i}{\partial\dot{x}^i\,\partial x^m}\dot{x}^m + 2\frac{\partial^2\Gamma^i}{\partial\dot{x}^i\,\partial\dot{x}^m}\Gamma^m - \frac{\partial\Gamma^i}{\partial\dot{x}^m}\frac{\partial\Gamma^m}{\partial\dot{x}^i} \tag{9.19}$$

transforms into

$$2\hat{B} = 2B + (n-1)\left(\frac{\partial\psi}{\partial x^i}\dot{x}^i - 2\frac{\partial\psi}{\partial\dot{x}^i}\Gamma^i + \psi^2\right) \tag{9.20}$$

by the changes (9.5) of the functions Γ^i. Hence we see from (9.11) that the scalar

$$^*B = B + M(n-1)\Gamma^0 \tag{9.21}$$

which is homogeneous of the second degree in the $\dot{x}^i$ remains unaltered by the transformations (9.5). The postulate $^*B = 0$ gives, therefore,

$$\Gamma^0 = -\frac{1}{M(n-1)}B. \tag{9.22}$$

If we introduce this value of Γ^0 in (9.9) this equation defines the *preferred projective normal parameter* up to linear fractional transformations.

---

[33] In the case of §5, we have

$$2B = B_{jk}\dot{x}^j\dot{x}^k, \qquad 2^*B = {}^*B_{jk}\dot{x}^j\dot{x}^k.$$

These equations remain valid for a general geometry of paths if we put, for the B_{jk}, $^*B_{jk}$, the values (9.25), (9.27) respectively.

In the case of the general projective geometry of paths, the Weyl projective curvature tensor has the components

$$
\begin{aligned}
W^i_{jkl} = {} & B^i_{jkl} + \frac{1}{n^2-1}\left[(nB_{jl} + B_{lj})\,\delta^i_k - (nB_{jk} + B_{kj})\,\delta^i_l\right] \\
& + \frac{1}{n+1}(B_{kl} - B_{lk})\,\delta^i_j \\
& + \dot{x}^p \frac{\partial}{\partial \dot{x}^j}\left[\frac{1}{n^2-1}(B_{lp} - B_{pl})\,\delta^i_k - \frac{1}{n^2-1}(B_{kp} - B_{pk})\,\delta^i_l\right. \\
& \left. + \frac{1}{n+1}(B_{kl} - B_{lk})\,\delta^i_p\right],
\end{aligned}
\tag{9.23}
$$

where

$$
\begin{aligned}
B^i_{jkl} = {} & \left(\frac{\partial^3 \Gamma^i}{\partial \dot{x}^j\,\partial \dot{x}^k\,\partial x^l} - \frac{\partial^3 \Gamma^i}{\partial \dot{x}^j\,\partial \dot{x}^k\,\partial \dot{x}^m}\frac{\partial \Gamma^m}{\partial \dot{x}^l}\right) - \left(\frac{\partial^3 \Gamma^i}{\partial \dot{x}^j\,\partial \dot{x}^l\,\partial x^k} - \frac{\partial^3 \Gamma^i}{\partial \dot{x}^j\,\partial \dot{x}^l\,\partial \dot{x}^m}\frac{\partial \Gamma^m}{\partial \dot{x}^k}\right) \\
& + \frac{\partial^2 \Gamma^i}{\partial \dot{x}^m\,\partial \dot{x}^l}\frac{\partial^2 \Gamma^m}{\partial \dot{x}^j\,\partial \dot{x}^k} - \frac{\partial^2 \Gamma^i}{\partial \dot{x}^m\,\partial \dot{x}^k}\frac{\partial^2 \Gamma^m}{\partial \dot{x}^j\,\partial \dot{x}^l}
\end{aligned}
\tag{9.24}
$$

and

$$
B_{jk} = B^i_{jki}. \tag{9.25}
$$

The *curvature tensor of the projective geometry of paths* under consideration is

$$
{}^*B^i_{jkl} = W^i_{jkl} + \frac{{}^*B_{jk}}{n-1}\,\delta^i_l - \frac{{}^*B_{jl}}{n-1}\,\delta^i_k, \tag{9.26}
$$

where

$$
{}^*B_{ij} = \frac{\partial^2\,{}^*B}{\partial \dot{x}^i\,\partial \dot{x}^j}. \tag{9.27}
$$

From (9.26) and (9.23), it follows that

$$
{}^*B_{ij} = {}^*B^i_{jki}. \tag{9.28}
$$

Besides the tensor ${}^*B^i_{jkl}$, we have in the general projective geometry of paths still the tensor[34]

$$
\Pi^i_{jkl} = \frac{\partial^3}{\partial \dot{x}^j\,\partial \dot{x}^k\,\partial \dot{x}^l}\left(\Gamma^i - \frac{1}{n+1}\frac{\partial \Gamma^m}{\partial \dot{x}^m}\dot{x}^i\right) \tag{9.29}
$$

and the tensor

$$
{}^*B_{jkl} = \frac{\partial^3\,{}^*B}{\partial \dot{x}^j\,\partial \dot{x}^k\,\partial \dot{x}^l} = \frac{\partial\,{}^*B_{jk}}{\partial \dot{x}^l} \tag{9.30}
$$

[34] J. Douglas, l.c., p. 157, equation (5.11).

which are of importance. The equation

$$\Pi^i_{jkl} = 0 \tag{9.31}$$

is a necessary and sufficient condition that a general projective geometry of paths have the same paths as a restricted projective geometry of paths,[35] the equations

$$\Pi^i_{jkl} = 0, \qquad {}^*B_{jkl} = 0 \tag{9.32}$$

are necessary and sufficient conditions that it be itself a restricted projective geometry of paths.[36] Therefore (§7), the equations

$$\Pi^i_{jkl} = 0, \qquad {}^*B^i_{jkl} = 0, \tag{9.33}$$

from the second of which ${}^*B_{jkl} = 0$ is a consequence, are, for $n > 2$, necessary and sufficient conditions that the class K of affine connections belonging to the system (9.1) of paths contain an euclidean-affine connection and that the affine parameter on the paths corresponding to this connection be a projective normal parameter.

For sake of brevity, we do not enter into the generalization, to the general geometry of paths, of the method set forth in §6, a generalization which offers no difficulty.

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[35] J. Douglas, l.c., p. 157.

[36] For the proof of this theorem, the generalization of the method of §6 would be necessary.

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SUR LES ENSEMBLES QUASI-MINIMAUX DANS LES SYSTÈMES DYNAMIQUES

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Introduction

Considérons un système dynamique, dont le mouvement est donné par les équations différentielles:

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n) \qquad (i = 1, 2, \cdots, n)$$

où $x_1, \cdots, x_n$ sont les paramètres qui déterminent la position du système et où t désigne le temps.

Nous considérons l'ensemble des valeurs $x_1, \cdots, x_n$ comme un point p de coordonnées $(x_1, \cdots, x_n)$ dans l'espace Euclidien E^n à n dimensions. Les mouvements des systèmes dynamiques seront caractérisés par la position des points sur les trajectoires décrites par ces points $p \subset E^n$: nous désignerons la position du point p sur la trajectoire à l'instant t par $f(p, t)$; $f(p, o) = p$.

Supposons que les fonctions X_i dans les seconds membres des équations différentielles soient continues par rapport à tous les arguments et qu'elles satisfassent à la condition de Lipschitz dans un certain domaine fermé D. Ce n'est que sous certaines restrictions que nous aurons le théorème de l'existence et de l'unicité des solutions ainsi que le théorème de la dépendance continue des trajectoires des conditions initiales.

Ce dernier théorème peut être énoncé de la manière suivante: Soit p un point fixe et q un point arbitraire de l'espace E^n; alors pour tout $\epsilon > 0$ et pour tout nombre positif T arbitrairement grand, on peut indiquer un nombre $\delta > 0$ assez petit pour qu'on ait $\rho(f(p, t), f(q, t)) < \epsilon$ pour toute valeur de t, appartenant à l'intervalle $0 \leqq t \leqq T$ lorsque $\rho(p, q) < \delta$; $\rho(x, y)$ désignant la distance entre les points x et y dans E^n.

Nous considérons dans la suite les ensembles invariants, définis de la manière suivante: l'ensemble $M \subset E^n$ est dit *invariant* si $f(p, t) \subset M$, pour tout $p \subset M$.

Dans la dynamique générale une place de grande importance revient à certains mouvements qui représentent une généralisation des mouvements périodiques. Ce sont les mouvements *stables au sens de Poisson* et comme cas particulier, les mouvements *récurrents* de Birkhoff. La trajectoire $f(p, t)$ (p étant fixé) est dite *stable au sens de Poisson* si pour tout nombre positif ϵ et pour tout nombre positif T arbitrairement grand il existe des valeurs de t—tant positives que négatives et surpassant T en valeur absolue, pour lesquelles on a:

$$\rho(p, f(p, t)) < \epsilon.$$

Un mouvement stable au sens de Poisson est dit *récurrent* si à tout $\epsilon > 0$ il correspond un nombre positif constant $\tau = \tau(\epsilon)$ tel que dans l'intervalle de temps de longueur τ tout point du mouvement $f(p, t)$ passe à une distance inférieure à ϵ de tout point de la trajectoire du mouvement considéré.

Les mouvements récurrents ont été introduits et étudiés par Birkhoff. Les principaux résultats de ses études peuvent être résumés comme il suit:[1]

Un ensemble invariant, compact et fermé Σ est dit *minimal*, s'il ne peut contenir un sous-ensemble différant de Σ et jouissant des mêmes propriétés. Birkhoff a démontré que tout ensemble invariant, compact et fermé renferme au moins un ensemble minimal. De plus les recherches de Birkhoff ont établi que les ensembles minimaux sont intimement liés aux mouvements récurrents: tous les mouvements d'un ensemble minimal sont récurrents et tout mouvement récurrent est contenu dans un ensemble minimal. La trajectoire de tout mouvement récurrent est dense partout dans l'ensemble minimal qui contient le mouvement.

Dans cet article j'introduis une notion nouvelle. Je désigne un ensemble invariant compact et fermé comme un ensemble *dynamiquement indécomposable*, s'il ne peut être représenté comme une somme de deux ensembles qui soient différents de l'ensemble donné et qui jouissent des mêmes propriétés, c'est à dire qui soient invariants et fermés. Les ensembles dynamiquement indécomposables présentent une généralisation de la notion des ensembles minimaux, ceux-ci étant dynamiquement indécomposables. Les ensembles dynamiquement indécomposables forment deux classes distinctes, celle des *ensembles indécomposables ordinaires* et celle des *ensembles indécomposables exceptionnels*. Je nomme ensuite les ensembles ordinaires indécomposables—ensembles quasi-minimaux, car ce sont précisement ceux qui représentent une telle généralisation des ensembles minimaux qui permettent d'inclure les mouvements stables au sens de Poisson.

Le rôle des ensembles quasi-minimaux dans la dynamique est caractérisé par les thèses suivantes qui forment le résultat principal de ce mémoire: *tout ensemble stable au sens de Poisson appartient à un ensemble quasi-minimal, la trajectoire d'un tel mouvement étant partout dense dans cet ensemble, et tout mouvement quasi-minimal contient un mouvement stable au sens de Poisson, dont la trajectoire est dense partout dans cet ensemble.*

1

Nous dirons que l'ensemble invariant, compact et fermé Θ des points du système dynamique est *dynamiquement indécomposable*, s'il ne peut être représenté comme la somme de deux ensembles différents de Θ qui jouissent des mêmes propriétés, c'est à dire qui soient fermés et invariants.

THÉORÈME I—*Soit Θ un ensemble dynamiquement indécomposable et G un domaine relatif arbitraire de cet ensemble; alors l'ensemble des points appartenant aux trajectoires qui traversent le domaine G est partout dense dans Θ.*

[1] Birkhoff: *Dynamical systems.* N.Y. 1927 Ch. VII.

Admettons que le théorème ne soit pas vrai. Alors on pourrait indiquer dans l'ensemble Θ deux domaines relatifs G et G' tels qu'aucune trajectoire traversant le domaine G ne croise le domaine G'. Considérons l'ensemble des points, appartenant aux trajectoires qui traversent le domaine G et désignons par Θ_1 la fermeture de cet ensemble. L'ensemble Θ_1 obtenu sera fermé, de plus, il est invariant, comme la fermeture d'un ensemble invariant.

Il est évident que $\Theta_1 \neq \Theta$, car aucun point du domaine $G' \subset \Theta$ n'appartient à l'ensemble Θ_1.

Considérons ensuite l'ensemble $\Theta - \Theta_1$ et soit Θ_2 la fermeture de cet ensemble. Il est évident que $\Theta_2 \neq \Theta$ car aucun point du domaine $G \subset \Theta$ n'appartient à l'ensemble Θ_2.

Nous avons donc

$$\Theta = \Theta_1 + \Theta_2,$$

Θ_1 et Θ_2 étant fermés et distincts de Θ.

Le résultat obtenu contrédit la supposition que l'ensemble Θ est dynamiquement indécomposable, et le théorème est démontré.

THÉORÈME II—*Tout ensemble dynamiquement indécomposable* Θ *contient un mouvement* $f(r, t)$ *tel que sa trajectoire est partout dense dans cet ensemble.*

Soit G un domaine relatif arbitraire dans l'ensemble Θ. Donnons nous un nombre arbitraire $\epsilon_1 > 0$ et montrons qu'il existe un domaine $G_1 \subset G$ tel que toute trajectoire croisant le domaine G_1, passe à une distance inférieure à ϵ_1 de tout point de l'ensemble Θ.

Recouvrons l'ensemble Θ par un nombre fini de domaines, dont les diamètres ne surpassent pas ϵ_1, ce qui est possible l'ensemble Θ étant compact.

Soient $G^1, G^2, \cdots, G^N$ ces domaines, rangés dans un ordre arbitraire.

Quel que soit le domaine G, il se trouvera toujours, en vertu du théorème précédent, un point $p \subset G$ tel que la trajectoire déterminée par ce point $f(p, t)$ croisera le domaine G^1. Par conséquent on peut indiquer un moment du temps t_1 pour lequel $f(p, t_1) \subset G^1$. En vertu du théorème de la dépendance continue des trajectoires du point initial on peut décrire autour du point p un voisinage $g' \subset g$ si petit que $f(g', t_1) \subset G^1$.

Il est évident que toutes les trajectoires qui sortent du domaine G' traversent le domaine G^1. De même on peut choisir dans le domaine G' un point p' tel que la trajectoire que ce point détermine, traverse le domaine G^2 et on peut indiquer un moment du temps t_2 pour lequel $f(p', t_2) \subset G^2$. Décrivons autour du point p' un voisinage $g'' \subset g'$ si petit que $f(g'', t_2) \subset g^2$. Toutes les trajectoires qui sortent du domaine G'' traverseront le domaine G' et G^2.

En poursuivant ces raisonnements jusqu'à ce que tous les domaines $G^1, G^2, \cdots, G^N$, soient épuisés nous obtenons un système décroissant de domaines: $G \supset G' \supset G'' \supset \cdots \supset G^{(N)}$. Nous le compléterons par le choix d'un domaine G_1 assez petit pour que sa fermeture $\bar{G}_1 \subset G^{(N)} \subset G$. Toute trajectoire croisant le domaine G_1, passera à une distance inférieure à ϵ de tout point de l'ensemble Θ.

Soient $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ une suite de nombres positifs convergeant vers zéro. En appliquant le procédé décrit plus haut nous pouvons construire conformément aux nombres de cette suite une suite de domaines: $G_1, G_2, \cdots, G_n, \cdots$ telles que: 1) la fermeture de chaque domaine est contenue dans les domaines précédants; 2) toute trajectoire traversant le domaine G_k passe à une distance inférieure à ϵ_k de tout point de l'ensemble Θ.

Une telle suite de domaines a une partie commune qui n'est pas vide. La trajectoire $f(r, t)$ déterminée par tout point r appartenant à cette partie commune est partout dense dans Θ, car elle est fait partie des faisceaux de trajectoires qui sortent de ces domaines G_k.

2

Considérons un ensemble Θ dynamiquement indécomposable. Désignons par Θ_0 l'ensemble des points des trajectoires des mouvements périodiques et des points de repos. Nous dirons que l'ensemble Θ est un ensemble indécomposable *ordinaire* si pour tout $\epsilon > 0$ le ϵ-voisinage de chaque point $p \subset \Theta - \Theta_0$ contient des points de Θ qui n'appartiennent pas à la trajectoire du mouvement $f(p, t)$.

Dans le cas où on peut indiquer un tel point $p \subset \Theta - \Theta_0$ et un nombre positif ϵ assez petit pour que le ϵ-voisinage du point p ne contienne que des points de Θ appartenant à la trajectoire du mouvement $f(p, t)$ nous dirons que l'ensemble Θ est un ensemble *exceptionnel* et que la trajectoire $f(p, t)$ est une trajectoire *caractéristique* pour cet ensemble exceptionnel.

Considérons les exemples suivants, comme exemples simples d'ensembles exceptionnels.

Exemple 1.—Examinons les mouvements des points d'un espace Euclidien à deux dimensions E^2, déterminé par les équations différentielles:

$$\frac{dx}{dt} = -y\{x^2 + (1 - y)^2\}$$

$$\frac{dy}{dt} = x\{x^2 + (1 - y)^2\}$$

où x et y sont des coordonnées cartésiennes du point. Les trajectoires de ce système dynamique seront constituées du point de repos à l'origine, d'un ensemble de circonférences ayant leur centre commun à l'origine et de rayons $r \neq 1$, et des trajectoires situées sur la circonférence $x^2 + y^2 = 1$. Les point de cette dernière circonférence se partagent en deux ensembles appartenant a deux mouvements différents: l'un est le point de repos avec les coordonnées (0, 1) l'autre a pour trajectoire les autres points de la circonférence.

L'ensemble formé par les deux trajectoires sera un ensemble indécomposable et exceptionnel, et la seconde trajectoire une trajectoire caractéristique.

Exemple 2.—Supposons que les mouvements des points de l'espace E^2 sont déterminés par les équations différentielles:

$$\frac{d\rho}{dt} = -\rho \ln \rho$$

$$\frac{d\varphi}{dt} = \frac{1}{a}$$

où ρ et φ sont les coordonnées polaires du point mobile et où a est une constante quelconque.

Les équations des trajectoires seront:

$$\rho = e^{Ce^{-a\varphi}},$$

où C est la constante d'intégration.

L'ensemble des trajectoires du système dynamique aura l'aspect suivant: à l'origine des coordonnées se trouve un point de repos. La circonférence $\rho = 1$ sera la trajectoire du mouvement périodique. Les points situés dans la couronne $0 < \rho < 1$ décrivent des spirales se rapprochant asymptotiquement de la circonférence $\rho = 1$ pour $t \to +\infty$ et de l'origine pour $t \to -\infty$.

Examinons l'ensemble formé par les trois trajectoires suivantes: par le point de repos, par la circonférence et par la spirale décrite par un point quelconque situé à l'intérieur de cette circonférence. Cet ensemble sera un ensemble indécomposable et exceptionnel, la spirale étant sa trajectoire caractéristique.

THÉORÈME III. *Un ensemble indécomposable exceptionnel* Θ *ne contient qu'un seul mouvement tel que sa trajectoire est partout dense dans* Θ, *cette trajectoire étant la trajectoire caractéristique.*

En effet, soit $f(r, t)$ la trajectoire d'un mouvement caractéristique de l'ensemble Θ. Décrivons autour du point r un voisinage du rayon ϵ assez petit pour que le ϵ-voisinage du point r ne contienne que des points appartenant à la trajectoire $f(r, t)$.

L'ensemble Θ contient immanquablement un mouvement dont la trajectoire est partout dense dans cet ensemble. En supposant que cette trajectoire diffère de $f(r, t)$, nous arrivons à la conclusion que le point r ne sera pas son point limite, ce qui est une contradiction. Nous aboutissons également à une contradiction en admettant qu'il existe à part le mouvement $f(r, t)$ une autre mouvement, dont la trajectoire est partout dense dans Θ. Le théorème est démontré.

THÉORÈME IV. *Aucune semi-trajectoire du mouvement caractéristique* $f(r, t)$ *de l'ensemble exceptionnel dynamiquement indécomposable* Θ *ne peut être partout dense dans cet ensemble.*

Admettons que l'une des semi-trajectoires du mouvement $f(r, t)$, par exemple la semi-trajectoire positive, soit partout dense dans l'ensemble Θ. Dans ce cas tous les points de la semi-trajectoire négative du mouvement $f(r, t)$ seront des points limites ω de sa trajectoire positive. Mais alors en vertu du théorème de la dépendance continue des trajectoires des condition initiales, les points de toute la trajectoire du mouvement $f(r, t)$ seront des points limites ω de la trajectoire positive de ce même mouvement.

D'un autre côté, la trajectoire $f(r, t)$ étant une trajectoire caractéristique, on peut indiquer un $\epsilon > 0$ tel que le ϵ-voisinage du point r ne contienne que des points appartenant à la trajectoire $f(r, t)$. Construisons dans le point r l'élément d'un plan perpendiculaire à la trajectoire $f(r, t)$. Décrivons dans ce plan autour du point r un vosinage sphérique fermé (par rapport à l'élément du plan) avec un rayon qui ne surpasse pas ϵ et qui en outre soit assez petit pour que ce voisinage, avec les points de sa frontière, appartienne à l'élément considéré.

Examinons l'ensemble dénombrable des points d'intersection de la semi-

trajectoire positive du mouvement $f(r, t)$ avec le ϵ-voisinage que nous venons de construire. En vertu de nos suppositions tout point de cet ensemble sera un point limit ω du mouvement $f(r, t)$ et par conséquent l'ensemble lui-même sera un ensemble parfait. Mais alors il ne peut pas être un ensemble dénombralle. Nous arrivons à une contradiction et le théorème est démontré.

THÉORÈME V. *Soit Θ un ensemble ordinaire dynamiquement indécomposable et soit $f(r, t)$ une trajectoire partout dense dans cet ensemble. Désignons par Θ_α et Θ_ω l'ensemble des points limites α resp. ω de la trajectoire $f(r, t)$; alors $\Theta = \Theta_\alpha + \Theta_\omega$.*

Admettons que le théorème ne soit pas vrai. Alors l'ensemble $\Theta' = \Theta - (\Theta_\alpha + \Theta_\omega)$ n'est pas vide. On sait que les ensembles Θ_α et Θ_ω, et par conséquent leur somme sont des ensembles fermés. Cela étant, l'ensemble Θ' sera un ensemble ouvert relativement à Θ. Par conséquent il se trouvera un domaine $G = \Theta'$ qui ne renfermera pas des points des ensembles Θ_α et Θ_ω.

L'ensemble Θ étant un ensemble ordinaire on peut indiquer un point $p \subset G$ qui ne sera pas situé sur la trajectoire du mouvement $f(r, t)$. Le point p n'appartiendra évidemment pas au points limites α ou ω du mouvement $f(r, t)$. Cela étant, on peut décrire autour du point p un voisinage $G' \subset G$ d'un diamètre assez petit pour qu'il ne contienne aucun point de la trajectoire du mouvement $f(r, t)$. Mais ceci contredit la supposition que la trajectoire du mouvement $f(r, t)$ est partout dense dans Θ. Le théorème est démontré.

THÉORÈME VI. *Soit Θ un ensemble ordinaire dynamiquement indécomposable et $f(r, t)$ une trajectoire partout dense dans cet ensemble. Alors au moins l'une des semi-trajectoires de ce mouvement sera partout dense dans Θ.*

Admettons de nouveau que ce théorème ne soit vrai, c'est à dire qu'aucune des semi-trajectoires $f(r, t)$ ne soit partout dense dans Θ. Désignons respectivement par Θ_α et Θ_ω les ensembles des points limites α et ω du mouvement $f(r, t)$.

Les ensembles Θ_α et Θ_ω sont des ensembles fermés invariants qui, selon nos hypothèses, seront les parties vraies de l'ensemble Θ. De plus en vertu du théorème précédent $\Theta = \Theta_\alpha + \Theta_\omega$.

Les conclusions obtenues contredisent l'indécomposabilité de l'ensemble Θ, et le théorème est démontré.

3

Nous dirons qu'un ensemble invariant, compact et fermé est un ensemble *quasi-minimal* et nous le désignerons par le signe Ω si cet ensemble est dynamiquement indécomposable ordinaire.

THÉORÉME VII. *Tout mouvement stable au sens de Poisson $f(p, t)$ dont la trajectoire est compacte, appartient à un ensemble quasi-minimal Ω, la trajectoire de ce mouvement étant partout dense dans Ω.*

Désignons par Ω l'ensemble des points appartenant à la fermeture de la trajectoire $f(p, t)$. L'ensemble Ω est invariant, compact et fermé; de plus la trajectoire du mouvement $f(p, t)$ est partout dense dans Ω.

L'ensemble Ω est un ensemble quasi-minimal; en effet tout sous-ensemble

invariant fermé contenant $f(p, t)$ coïncide necessairement avec Ω, donc Ω est indécomposable.

Il nous reste à montrer que l'ensemble Ω ne peut pas être un ensemble exceptionnel. En effet si cet ensemble était exceptionnel, alors en vertu du théorème IV la trajectoire $f(p, t)$ serait une trajectoire caractéristique et aucune de ses semi-trajectoires ne serait partout dense dans Ω. Mais ceci contredit la définition de la stabilité au sens de Poisson. Le théorème est démontré.

THÉORÈME VIII. *Tout ensemble quasi-minimal Ω contient un mouvement stable au sens de Poisson $f(p, t)$, dont la trajectoire est partout dense dans cet ensemble.*

L'ensemble Ω étant indécomposable, il en découle l'existence d'un mouvement $f(r, t)$, dont la trajectoire est partout dense dans Ω. L'ensemble Ω étant un ensemble ordinaire, au moins l'une des semi-trajectoires du mouvement $f(r, t)$ sera partout dense dans cet ensemble; admettons pour fixer les idées que cela soit la semi-trajectoire positive.

Assignons nous un nombre positif arbitraire ϵ_1 et montrons que dans tout domaine relatif $G_0 \subset \Omega$ il se trouvera un domaine G_1, tel que tout point du domaine G_1 ayant quitté le domaine G_0, passera au moins une fois à une distance inférieure à ϵ_1 de tout point de l'ensemble Ω et retournera ensuite dans le domaine G_0, le temps variant dans un sens positif ou négatif.

Recouvrons l'ensemble Ω par un nombre fini de domaines $G^1, G^2, \cdots, G^N$ dont les diamètres ne surpassent pas ϵ_1. Considérons le système des domaines $G^1, G^2, \cdots, G^N, G^{N+1}$ où $G^{N+1} = G_0$.

Des propriétés du mouvement $f(r, t)$, dont la trajectoire est partout dense dans Ω, découlent la possibilité de choix du point initial r tel que $r \subset G$, et l'existence d'une suite finie de nombres positifs croissants $t_1, t_2, \cdots, t_N, t_{N+1}$ telle que

$$f(r, t_i) \subset G^i \qquad (i = 1, 2, \cdots N, N + 1).$$

En vertu du théorème sur la dépendance continue des trajectoires des conditions initiales, on peut décrire autour du point r un domaine $G' \subset G_0$ assez petit pour que

$$f(G', t_i) \subset G^i \qquad (i = 1, 2, \cdots, N, N + 1).$$

Alors, le temps variant dans le sens positif, tout point du domaine G' passera au moins une fois à une distance inférieure à ϵ_1 de tout point de l'ensemble Ω et retournera ensuite dans le domaine G_0.

Considérons le domaine $G'' = f(G', t_{N+1}) \subset G_0$. Tout point de ce domaine jouit des mêmes propriétés que les points du domaine G', mais seulement pour le temps variant dans le sens négatif.

Nous pouvons appliquer à présent au domaine G'' les raisonnements que nous avons faits plus haut par rapport au domaine G_0, et construire un domaine $G_1 \subset G'' \subset G_0$, tout point duquel passera au moins une fois à une distance inférieure à ϵ_1 de tout point de l'ensemble Ω et retournera ensuite dans G_0.

Prenant en considération que $G_1 \subset G''$ nous arrivons à la conclusion que tout

trajectoire positive du mouvement $f(r, t)$ avec le ϵ-voisinage que nous venons de construire. En vertu de nos suppositions tout point de cet ensemble sera un point limit ω du mouvement $f(r, t)$ et par conséquent l'ensemble lui-même sera un ensemble parfait. Mais alors il ne peut pas être un ensemble dénombralle. Nous arrivons à une contradiction et le théorème est démontré.

THÉORÈME V. *Soit Θ un ensemble ordinaire dynamiquement indécomposable et soit $f(r, t)$ une trajectoire partout dense dans cet ensemble. Désignons par Θ_α et Θ_ω l'ensemble des points limites α resp. ω de la trajectoire $f(r, t)$; alors $\Theta = \Theta_\alpha + \Theta_\omega$.*

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L'ensemble Θ étant un ensemble ordinaire on peut indiquer un point $p \subset G$ qui ne sera pas situé sur la trajectoire du mouvement $f(r, t)$. Le point p n'appartiendra évidemment pas au points limites α ou ω du mouvement $f(r, t)$. Cela étant, on peut décrire autour du point p un voisinage $G' \subset G$ d'un diamètre assez petit pour qu'il ne contienne aucun point de la trajectoire du mouvement $f(r, t)$. Mais ceci contredit la supposition que la trajectoire du mouvement $f(r, t)$ est partout dense dans Θ. Le théorème est démontré.

THÉORÈME VI. *Soit Θ un ensemble ordinaire dynamiquement indécomposable et $f(r, t)$ une trajectoire partout dense dans cet ensemble. Alors au moins l'une des semi-trajectoires de ce mouvement sera partout dense dans Θ.*

Admettons de nouveau que ce théorème ne soit vrai, c'est à dire qu'aucune des semi-trajectoires $f(r, t)$ ne soit partout dense dans Θ. Désignons respectivement par Θ_α et Θ_ω les ensembles des points limites α et ω du mouvement $f(r, t)$.

Les ensembles Θ_α et Θ_ω sont des ensembles fermés invariants qui, selon nos hypothèses, seront les parties vraies de l'ensemble Θ. De plus en vertu du théorème précédent $\Theta = \Theta_\alpha + \Theta_\omega$.

Les conclusions obtenues contredisent l'indécomposabilité de l'ensemble Θ, et le théorème est démontré.

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Nous dirons qu'un ensemble invariant, compact et fermé est un ensemble *quasi-minimal* et nous le désignerons par le signe Ω si cet ensemble est dynamiquement indécomposable ordinaire.

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L'ensemble Ω est un ensemble quasi-minimal; en effet tout sous-ensemble

invariant fermé contenant $f(p, t)$ coïncide necessairement avec Ω, donc Ω est indécomposable.

Il nous reste à montrer que l'ensemble Ω ne peut pas être un ensemble exceptionnel. En effet si cet ensemble était exceptionnel, alors en vertu du théorème IV la trajectoire $f(p, t)$ serait une trajectoire caractéristique et aucune de ses semi-trajectoires ne serait partout dense dans Ω. Mais ceci contredit la définition de la stabilité au sens de Poisson. Le théorème est démontré.

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Assignons nous un nombre positif arbitraire ϵ_1 et montrons que dans tout domaine relatif $G_0 \subset \Omega$ il se trouvera un domaine G_1, tel que tout point du domaine G_1 ayant quitté le domaine G_0, passera au moins une fois à une distance inférieure à ϵ_1 de tout point de l'ensemble Ω et retournera ensuite dans le domaine G_0, le temps variant dans un sens positif ou négatif.

Recouvrons l'ensemble Ω par un nombre fini de domaines $G^1, G^2, \cdots, G^N$ dont les diamètres ne surpassent pas ϵ_1. Considérons le système des domaines $G^1, G^2, \cdots, G^N, G^{N+1}$ où $G^{N+1} = G_0$.

Des propriétés du mouvement $f(r, t)$, dont la trajectoire est partout dense dans Ω, découlent la possibilité de choix du point initial r tel que $r \subset G$, et l'existence d'une suite finie de nombres positifs croissants $t_1, t_2, \cdots, t_N, t_{N+1}$ telle que

$$f(r, t_i) \subset G^i \qquad (i = 1, 2, \cdots N, N + 1).$$

En vertu du théorème sur la dépendance continue des trajectoires des conditions initiales, on peut décrire autour du point r un domaine $G' \subset G_0$ assez petit pour que

$$f(G', t_i) \subset G^i \qquad (i = 1, 2, \cdots, N, N + 1).$$

Alors, le temps variant dans le sens positif, tout point du domaine G' passera au moins une fois à une distance inférieure à ϵ_1 de tout point de l'ensemble Ω et retournera ensuite dans le domaine G_0.

Considérons le domaine $G'' = f(G', t_{N+1}) \subset G_0$. Tout point de ce domaine jouit des mêmes propriétés que les points du domaine G', mais seulement pour le temps variant dans le sens négatif.

Nous pouvons appliquer à présent au domaine G'' les raisonnements que nous avons faits plus haut par rapport au domaine G_0, et construire un domaine $G_1 \subset G'' \subset G_0$, tout point duquel passera au moins une fois à une distance inférieure à ϵ_1 de tout point de l'ensemble Ω et retournera ensuite dans G_0.

Prenant en considération que $G_1 \subset G''$ nous arrivons à la conclusion que tout

point du domaine $G_1 \subset G_0$ passera au moins une fois à une distance inférieure à ϵ_1 de tout point de l'ensemble Ω et retournera ensuite dans le domaine G_0 pour le temps variant dans les deux sens.

Soit $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ une suite de nombres positifs convergeant vers zéro. Conformément aux nombres de cette suite construisons dans Ω une suite de domaines relatifs $G_0, G_1, G_2, \cdots, G_n, \cdots$ telles que: 1) la fermeture de tout domaine soit contenue dans le domaine précédent, 2) les diamètres des domaines convergent vers zéro et 3) tout point du domaine G_{k+1} ayant quitté sa position initiale, passera à une distance ne surpassant pas ϵ_{k+1} de tout point de l'ensemble Ω et retournera ensuite dans le domaine G_k, le temps variant dans le sens négatif ou dans le sens positif.

Le trajectoire $f(p, t)$ déterminée par le point p appartenant à l'intersection de la suite des domaines que nous avons construits, appartiendera à chacun des faisceaux des trajectoires sortant de ces domaines. Cela étant, le point p, en décrivant sa trajectoire, traversera le voisinage arbitrairement petit de tout point de l'ensèmble Ω et en particulier le voisinage arbitrairement petit de tout point de sa trajectoire, pour le temps variants dans le sens négatif ou dans le sens positif.

Ceci veut dire que le mouvement $f(p, t)$ dont la trajectoire est partout dense dans Ω est stable au sens de Poisson. Le théorème est démontré.

Pour conclure citons un exemple simple d'ensembles quasi-minimaux.

Introduisons sur la surface du tore des coordonnées orthogonales curvilignes: φ l'azimuth du plan méridien et ψ l'angle polaire dans le plan méridien avec le kôle dans le point d'intersection de ce plan avec l'axe du tore.

Considérons sur la surface du tore l'ensemble de mouvements détérminés par les équations différentielles:

$$\frac{d\varphi}{dt} = a, \qquad \frac{d\psi}{dt} = b, \tag{1}$$

où a et b sont des constantes.

Il est connu que a et b étant incommensurables les trajectoires ne sont pas fermées, et l'ensemble des points de chacune des trajectoires est partout dense sur la surface du tore. La surface du tore est un ensemble minimal, formé des points des trajectoires de l'ensemble des mouvements récurrents ayant la puissance du continu.

Nous aurons un exemple d'un ensemble quasi-minimal qui ne sera pas minimal si dans l'exemple examiné du système dynamique les équations différentielles du mouvement seront remplacées par les équations suivantes:

$$\frac{d\varphi}{dt} = a\left(\sin^2\frac{\varphi}{2} + \sin^2\frac{\psi}{2}\right) \qquad \frac{d\psi}{dt} = b\left(\sin^2\frac{\varphi}{2} + \sin^2\frac{\psi}{2}\right). \tag{2}$$

Dans ce cas le caractère des mouvements changera de la manière suivante: le mouvement du système (1) dont la trajectoire

$$\varphi = \frac{a}{b}\psi$$

passait par l'origine des coordonnées, se décomposera dans le système (2) en trois mouvements différents: 1) le point de repos à l'origine des coordonnées et 2) deux mouvements instables au sens de Poisson. Le caractère de tous les autres mouvements variera quant aux vitesses, cependant leurs trajectoires resteront les mêmes, elles seront partout denses sur la surface du tore. Ces mouvement ne seront plus des mouvements récurrents, ce seront des mouvements stables au sens le Poisson.

La surface du tore se trouvant être dans ce cas un ensemble quasi-minimal ne sera pas un ensemble minimal car elle contient le point de repos comme sous-ensemble vrai, invariant et fermé.

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ÜBER DIVERGENZERSCHEINUNGEN DER LAGRANGESCHEN INTERPOLATIONSPOLYNOME STETIGER FUNKTIONEN

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Einleitung.

Es seien

$$x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)} \tag{1}$$

verschiedene im Intervall $-1 \leqq x \leqq +1$ liegende Werte und

$$y_1^{(n)}, y_2^{(n)}, \cdots, y_n^{(n)} \tag{2}$$

weitere n reelle Werte. Dann liefert das Polynom

$$L_n(x) = \sum_{k=1}^{n} y_k^{(n)} \frac{\omega_n(x)}{\omega_n'(x_k^{(n)})\,(x - x_k^{(n)})}, \tag{3}$$

$$\omega_n(x) = (x - x_1^{(n)})\,(x - x_2^{(n)}) \cdots (x - x_n^{(n)})$$

diejenige ganze rationale Funktion von höchstens $(n-1)$-tem Grade, welche an den Stellen (1), der Reihe nach, die Werte (2) annimmt. Wenn die Zahlen (2) die von der Funktion $f(x)$ an den Stellen (1) angenommenen Werte sind, so nennt man das Polynom (3) das Lagrangesche Interpolationspolynom der Funktion $f(x)$; dasselbe, wie es aus dem Früheren zu sehen ist, stimmt mit der Funktion an den Stellen (1) überein.

Wenn wir die Zahlen (1) bei $n = 1, 2, \cdots$ bilden, erhalten wir die Zahlenmenge

$$\begin{array}{l} x_1^{(1)} \\ x_1^{(2)}, x_2^{(2)} \\ x_1^{(3)}, x_2^{(3)}, x_3^{(3)} \\ \cdots\cdots\cdots\cdots\cdots \\ x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)} \\ \cdots\cdots\cdots\cdots\cdots \end{array}$$

die wir eine Grundpunktgruppe der Lagrangeschen Interpolation nennen; die Zahlen (1) sind die Grundpunkte der Interpolation.

Wir erhalten eine der interessantesten Grundpunktgruppen wenn wir als

Grundpunkte des Lagrangeschen Interpolationspolynoms n-ter Ordnung die Zahlen

$$x_1^{(n)} = \cos\frac{\pi}{2n}, \qquad x_2^{(n)} = \cos 3\frac{\pi}{2n}, \cdots,$$

(4)

$$x_k^{(n)} = \cos(2k-1)\frac{\pi}{2n}, \cdots, \qquad x_n^{(n)} = \cos(2n-1)\frac{\pi}{2n}$$

nämlich die Nullstellen des Tschebyscheffschen Polynoms n-ter Ordnung

$$T_n(x) = \cos(n \arccos x)$$

nehmen.

Vor kurzer Zeit habe ich gezeigt,[1] dass es für die Tschebyscheffschen Grundpunkte eine stetige Funktion $f(x)$ gibt, für welche die Folge der Lagrangeschen Interpolationspolynome

$$L_1[f(x)], L_2[f(x)], \cdots, L_n[f(x)], \cdots \tag{5}$$

im Intervall $-1 \leqq x \leqq +1$ fast überall divergiert, ja sogar fast überall unbeschränkt ist (also die Menge der etwaigen Konvergenzpunkte von Masse 0 ist).

In dieser Arbeit werde ich diesen Satz folgendermassen verallgemeinern: *Es seien die Grundpunkte der Lagrangeschen Interpolation die Zahlen*

$$x_k^{(n)} = \cos(2k-1)\frac{\pi}{2n} \qquad (k = 1, 2, \cdots, n)$$

d.h. die Nullstellen des Tschebyscheffschen Polynoms n-ter Ordnung

$$T_n(x) = \cos(n \arccos x)$$

dann gibt es eine stetige Funktion $f(x)$ deren Folge von Lagrangeschen Interpolationspolynomen in sämtlichen Punkten des Intervalls $-1 \leqq x \leqq +1$ divergiert, sogar unbeschränkt ist.

Die Methode, mit welcher wir diesen Satz beweisen, zeigt viele Ähnlichkeit mit der in der oben zitierten Arbeit benützten. Wir wollen uns aber nicht auf diese Arbeit stützen und so skizzieren wir auch hier ganz kurz die Methode.

Man nennt die Funktionen

$$l_k^{(n)}(x) = \frac{\omega_n(x)}{\omega_n'(x_k^{(n)})(x - x_k^{(n)})} \qquad (k = 1, 2, \cdots, n)$$

die Grundfuktionen der Interpolation. In der Untersuchung der Divergenzerscheinungen der Folge (5) spielen die Funktionen (die Summe der absoluten Beträge der Grundfunktionen)

$$\lambda_n(x) = \sum_{k=1}^{n} |l_k^{(n)}(x)| \tag{6}$$

[1] Géza Grünwald, "Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome," *Acta Litter. ac Scientiarum.* 7. (1935), S. 207–221.

eine entscheidende Rolle. Nämlich ein Satz von HAHN besagt,[2] dass wenn die Funktionen (6) in einem Punkte x_0 eine unbeschränkte Zahlenfolge bilden, dann gibt es eine im Intervalle $-1 \leqq x \leqq +1$ stetige Funktion deren Lagrangesche Interpolationspolynome im Punkte x_0 unbeschränkt sind.

Nun gilt im Falle Tschebyscheffscher Grundpunkte die Abschätzung

$$\lambda_n(x) \geqq \frac{1}{\pi} | T_n(x) | \log n - c_1(x) \tag{7}$$

wo $c_1(x)$ und in Folgenden $c_2(x), \cdots$ nur von x abhängige positive Zahlen bezeichnen. Aus dieser Ungleichung folgt, dass die Folge $\lambda_1(x), \lambda_2(x), \cdots, \lambda_n(x), \cdots$ in keinem Punkte des Intervalls $-1 \leqq x \leqq +1$ beschränkt sein kann, da sonst die Folge $| T_1(x) |, | T_2(x) |, \cdots, | T_n(x) |, \cdots$ zu Null konvergieren würde, was in Rücksicht auf $T_n(x) = \cos(n \text{ arc cos } x)$ unmöglich ist. Dieser Umstand ermöglicht uns in allen Punkten x_0 des Intervalls $-1 \leqq x \leqq +1$ eine stetige Funktion zu konstruieren, deren Lagrangesche Interpolationspolynome im Punkte x_0 unbeschränkt werden. Zum Beweis unseres Satzes benötigen wir anstatt (7) eine schärfere Abschätzung, die wir in §1 beweisen werden. Diese Ungleichung bezieht sich auf die Summe die aus (6) entsteht, indem man diejenigen Grundfunktionen $l_k^{(n)}(x)$ weglässt, die zu Abszissen $x_k^{(n)}$ gehören, welche bei einer m äquidistanten Teilung des Intervalls $-1 \leqq x \leqq +1$ im selben Teilintervall wie die Stelle x, oder in einem Teilintervall rechts davon liegen. Der Einfachheit halber behalten wir nur die Glieder mit ungeradem Index. Sie besitzen nämlich in diesem Fall konstantes Vorzeichen. Die präzise Form der Ungleichung

$$\lambda_n(x; m) \geqq \frac{1}{2\pi} | T_n(x) | \log m - c_3(x) \tag{8}$$

zeigt, dass die Abschätzung in der Anzahl m der Teilintervalle gleichmässig gilt und dieser Umstand bewirkt es, dass wir einen Divergenzsatz für sämtliche Stellen des Intervalls $-1 \leqq x \leqq +1$ erhalten. Wir gelangen zu unserem vollständigen Satz nach zwei Schritten. Zuerst in §3. beweisen wir das Vorhandensein einer stetigen Funktion $\varphi(x)$, deren Lagrangesche Interpolationspolynome in sämtlichen Punkten des Intervalls $-1 < x \leqq +1$ divergieren (sogar unbeschränkt werden). Der zweite wesentlich einfachere Schritt besteht darin, dass wir in §4. die Existenz einer stetigen Funktion $\psi(x)$ zeigen, deren Lagrangesche Interpolationspolynome in einem vorausgegebenen Punkte des Intervalls $-1 \leqq x \leqq +1$ unbeschränkt werden, aber in den anderen Punkten beschränkt bleiben. Es werden die beiden Funktionen $\varphi(x)$ und $\psi(x)$ effektiv gegeben. Die gesuchte stetige Funktion $f(x)$ wird durch die Summe von $\varphi(x)$ und $\psi(x)$ gebildet. In den bisher nicht erwähnten §2. beweisen wir zwei auf Tschebyscheffsche Grundpunkte bezügliche Hilfssätze.

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[2] HANS HAHN, „Über das Interpolationsproblem," *Math Zeitschrift, 1* (1918), S. 115–142.

1. **Untere Abschätzung geeigneter, aus den absoluten Beträgen der Grundfunktionen gebildeter Summen.**

Von nun an verstehen wir unter $x_k^{(n)}(k = 1, 2, \cdots, n)$ die Nullstellen des Tschebyscheffschen Polynoms n-ter Ordnung, dann lassen wir ebenso wie bei $l_k^{(n)}(x)$ die oberen Indizes fort. Auch schreiben wir x_k in der Form

$$x_k = \cos \vartheta_k, \qquad \vartheta_k = \frac{2k-1}{2n}\pi \qquad (k = 1, 2, \cdots, n).$$

Es ist offenbar

$$1 > x_1 > x_2 > \cdots > x_n > -1.$$

Es sei $-1 < x \leqq +1$; zerlegen wir das Intervall $-1 < x \leqq +1$ in m gleiche Teile; rechnen wir jeden Teil den rechten Endpunkt hinzu, den linken aber nicht. Es sei x im $(\mu + 1)$-ten Teilintervall enthalten (von links gerechnet) $\mu = 0, 1, 2, \cdots, m-1$. Betrachten wir diejenigen ungeraden Werte von k, für welche x_k im ersten, zweiten, $\cdots$, oder μ-ten Teilintervall liegt. Nehmen wir an, dass es solche Werte von k gibt; dazu genügt es, dass x_n und x_{n-1} zum ersten Teilintervall gehören, x aber nicht, also dass

$$1 + x_{n-1} \leqq \frac{2}{m} < 1 + x;$$

wegen $1 + x_{n-1} = \cos \vartheta_{n-1} - \cos \pi \leqq \pi - \vartheta_{n-1} < 6/n$ (wenn nämlich $\pi \geqq \alpha > \beta \geqq 0$, dann ist $\cos \beta - \cos \alpha < \alpha - \beta$ wie zum Beispiel aus der geometrischen Bedeutung des kosinus hervorgeht) genügt es also, wenn m die Ungleichung

$$\frac{2}{1+x} < m \leqq \frac{n}{3}$$

erfüllt, was wir voraussetzen werden.

Es sei also $2\nu + 1$ der kleinste ungerade Index, für welchen $x_{2\nu+1}$ in einem der ersten μ Intervalle liegt, und betrachten wir die Summe[3]

$$\lambda_n(x; m) = \sum_{k=\nu}^{\left[\frac{n-1}{2}\right]} |l_{2k+1}(x)|.$$

Im Falle Tschebyscheffscher Polynome

$$l_k(x) = l_k^{(n)}(x) = \frac{T_n(x)}{T_n'(x)(x - x_k)} = (-1)^{k+1}\frac{\sin \vartheta_k}{n}\frac{T_n(x)}{x - x_k},$$

also

$$l_{2k+1}(x) = \frac{\sin \vartheta_{2k+1}}{n}\frac{T_n(x)}{x - x_{2k+1}}$$

[3] $[\alpha]$ bedeutet die grösste ganze Zahl in α.

besitzt für die betrachteten Werte von k dasselbe Vorzeichen wie $T_n(x)$ woraus folgt, dass

$$\lambda_n(x;m) = \left| \sum_{k=\nu}^{\left[\frac{n-1}{2}\right]} \frac{\sin \vartheta_{2k+1}}{x - \cos \vartheta_{2k+1}} \frac{T_n(x)}{n} \right| = \left| \sum_{k=\nu}^{\left[\frac{n-1}{2}\right]} l_{2k+1}(x) \right|.$$

Da aber $\sin \vartheta/(x - \cos \vartheta)$ eine monoton abnehmende Funktion von ϑ ist, so ist

$$\lambda_n(x;m) = \frac{|T_n(x)|}{n} \sum_{k=\nu}^{\left[\frac{n-1}{2}\right]} \frac{\sin \vartheta_{2k+1}}{x - \cos \vartheta_{2k+1}} \geqq \frac{1}{2\pi} |T_n(x)| \int_{\vartheta_{2\nu+1}}^{\pi} \frac{\sin \vartheta}{x - \cos \vartheta} d\vartheta$$

$$= \frac{1}{2\pi} |T_n(x)| \left(\log \frac{1}{x - x_{2\nu+1}} - c_2(x) \right);$$

nun ist aber $\nu > 0$ (da x_1 wegen $1 - x_1 < \vartheta_1 = \pi/2n < 6/n \leqq 2/m$ gewiss im letzten Teilintervall liegt) also

$$0 < x - x_{2\nu+1} = (x - x_{2\nu-1}) + (x_{2\nu-1} - x_{2\nu+1}) < \frac{2}{m} + \vartheta_{2\nu+1} - \vartheta_{2\nu-1}$$

$$= \frac{2}{m} + \frac{2\pi}{n} < \frac{2}{m} + \frac{9}{n} \leqq \frac{5}{m};$$

daher gewinnt man wegen $|T_n(x)| \leqq 1$

$$\lambda_n(x;m) \geqq \frac{1}{2\pi} |T_n(x)| \left(\log \frac{m}{5} - c_2(x) \right) \geqq \frac{1}{2\pi} |T_n(x)| \log m - c_3(x).$$

Wählt man $c_3(x) = \frac{1}{2\pi}(c_2(x) + \log 5)$ so, dass $c_3(x) \geqq \frac{1}{2\pi} \log \frac{2}{1+x}$ ausfalle, so gilt diese Ungleichung auch für $m \leqq \frac{2}{1+x}$ falls dann unter $\lambda_n(x;m)$ Null zu verstehen ist; also gilt für $-1 \leqq x \leqq +1$, $m \leqq \frac{n}{3}$

$$\lambda_n(x;m) \geqq \frac{1}{2\pi} |T_n(x)| \log m - c_3(x). \tag{9}$$

2. Hilfssätze über Tschebyscheffsche Grundpunkte.

HILFSSATZ 1. *Es seien n und n' teilerfremde ungerade Zahlen. Ist ξ eine Tschebyscheffsche Abszisse zugleich von n-ter und n'-ter Ordnung d.h. eine Nullstelle von $T_n(x)$ und $T_{n'}(x)$, so ist $\xi = 0$. Ist ξ eine Tschebyscheffsche Abszisse zugleich von $2n$-ter und $2n'$-ter Ordnung, so ist entweder $\xi = 1/\sqrt{2}$, oder $\xi = -1/\sqrt{2}$. Endlich die Tschebyscheffschen Polynome n-ter und $2n'$-ter, ferner n'-ter und $2n$-ter Ordnung können keine gemeinsame Nullstellen haben.*

Im ersten Falle ist

$$\xi = \cos \frac{2k-1}{2n} \pi = \cos \frac{2k'-1}{2n'} \pi \qquad (1 \leqq k \leqq n,\ 1 \leqq k' \leqq n')$$

also ist

$$\frac{2k-1}{2n} = \frac{2k'-1}{2n'}$$

ein echter Bruch; auf reduzierte Form gebracht ist sein Nenner ein gemeinsamer Teiler von $2n$ und $2n'$ daher ist der Bruch gleich $\frac{1}{2}$, also

$$\xi = \cos \tfrac{1}{2}\pi = 0.$$

Im zweiten Falle ist

$$\xi = \cos \frac{2k-1}{4n}\pi = \cos \frac{2k'-1}{4n'}\pi \qquad (1 \leqq k \leqq 2n,\ 1 \leqq k' \leqq 2n')$$

also

$$\frac{2k-1}{4n} = \frac{2k'-1}{4n'}$$

wieder ein echter Bruch mit einem Nenner, der gemeinsamer Teiler von $4n$ und $4n'$ ist und offenbar nicht 2 sein kann; daher ist der Nenner 4, der Bruch selbst $\frac{1}{4}$ oder $\frac{3}{4}$, also

$$\xi = \cos \tfrac{1}{4}\pi = 1/\sqrt{2} \text{ oder } \xi = \cos \tfrac{3}{4}\pi = -1/\sqrt{2}.$$

Endlich wenn

$$\cos \frac{2k-1}{4n}\pi = \cos \frac{2k'-1}{2n'}\pi$$

oder

$$\cos \frac{2k-1}{2n}\pi = \cos \frac{2k'-1}{4n'}\pi$$

wäre, so müsste

$$\frac{2k-1}{4n} = \frac{2k'-1}{2n'} \text{ bezw. } \frac{2k-1}{2n} = \frac{2k'-1}{4n'}$$

sein, was offenbar unmöglich ist.

HILFSSATZ 2. *Es sei* $2\nu+1$ *der kleinste ungerade Index, für welchen*

$$2/\sqrt{n} > 1 + x_{2\nu+1}.$$

Dann ist

$$\left| \sum_{k=\nu}^{\left[\frac{n-1}{2}\right]} l_{2k+1}^{(n)}(-1) \right| > c \log n$$

wo $c > 0$ *eine absolute Konstante ist.*

Es ist in der Tat

$$\sum_{k=\nu}^{\left[\frac{n-1}{2}\right]} l_{2k+1}^{(n)}(-1) = \frac{1}{n} \sum_{k=\nu}^{\left[\frac{n-1}{2}\right]} \frac{\sin \vartheta_{2k+1}}{1+\cos \vartheta_{2k+1}} \geqq \frac{1}{2\pi} \int_{\vartheta_{2\nu+1}}^{\vartheta_{2\left[\frac{n-1}{2}\right]}} \frac{\sin \vartheta}{1+\cos \vartheta}\, d\vartheta$$

$$= \log \frac{1+\cos \vartheta_{2\nu+1}}{1+\cos \vartheta_{2\left[\frac{n-1}{2}\right]}} \geqq \log \frac{1/\sqrt{n}}{\pi/n} > c \log n.$$

3. Konstruktion einer stetigen Funktion, deren Interpolationspolynome im Intervall $-1 < x \leqq +1$ divergieren.

Es sei σ_n das Maximum der stetigen Funktion $\lambda_n(x)$ für $-1 \leqq x \leqq +1$ und $\rho(n)$ der grösste under den Werten $\sigma_1, \sigma_2, \cdots, \sigma_n$. Dann ist offenbar $\rho(1) \leqq \rho(2) \leqq \cdots \leqq \rho(n)$; ferner ist wegen $\lambda_n(x_k^{(n)}) = 1$, $\rho(n) \geqq \sigma_n \geqq 1$. Für eine beliebige Funktion $f(x)$, wenn $-1 \leqq f(x) \leqq +1$ und $\nu \leqq n$ ist, gilt für sämtliche Punkte des Intervalls $-1 \leqq x \leqq +1$ die Ungleichung

$$|L_\nu[f(x)]| \leqq \sum_{k=1}^{\nu} |l_k^{(\nu)}(x)| = \lambda_\nu(x) \leqq \sigma_\nu \leqq \rho(n).$$

Es sei $n_1, n_2, \cdots$ eine unendliche Folge natürlicher Zahlen mit $(n_i n_r) = 1$ $(i, r = 1, 2, \cdots)$. Von nun an wird jedes n aus dieser Folge gewählt. Es sei n_{i_1} eine beliebige Zahl der Folge $(i_1 > 1)$. Teilen wir nun das Intervall $-1 < x \leqq +1$ in i_1 gleiche Teile. Der rechte Endpunkt wird jedesmal dem Teilintervall zugerechnet, der linke aber nicht. Nach Hilfssatz 1. können die gemeinsamen Nullstellen von je zwei der Tschebyscheffschen Polynome $n_{i_1}, 2n_{i_1}, n_{i_1+1}, 2n_{i_1+1}, \cdots, n_{i_1+s}, 2n_{i_1+s}, \cdots, n_{2i_1}, 2n_{2i_1}$-ter Ordnung nur 0, $\pm 1/\sqrt{2}$ sein. Wir können also ein Polynom $\varphi_1(x)$ vom Grade n_{i_2} bestimmen,[4] das in den Nullstellen von ungeradem Index der Tschebyscheffschen Polynome n_{i_1+s} und $2n_{i_1+s}$-ter Ordnung (ausgenommen die etwaigen Nullstellen 0 und $\pm 1/\sqrt{2}$) wenn diese vom Endpunkte des s-ten Teilintervalls links liegen $(s = 1, 2, \cdots, i_1)$ den Wert $+1$ annimmt; in den übrigen Nullstellen der Tschebyscheffschen Polynome n_{i_1+s} und $2n_{i_1+s}$-ter Ordnung, eingenommen die Nullstellen 0 und $\pm 1/\sqrt{2}$, es sei $\varphi_1(x) = 0$ und im ganzen Intervall $-1 \leqq x \leqq +1$

$$|\varphi_1(x)| \leqq 2;$$

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[4] Es gilt nämlich der, mit dem Weierstrassschen Satz leicht beweisbare Satz: *Es sei $f(x)$ eine stetige Funktion im Intervall $-1 \leqq x \leqq +1$ und $x_1, x_2, \cdots, x_n$; $y_1, y_2, \cdots, y_n$ vorgegebene Zahlen die im Intervall $-1 \leqq x \leqq +1$ liegen, so gibt es zu jeden $\epsilon > 0$ ein Polynom $P(x)$, dass im Intervall $-1 \leqq x \leqq +1$*

$$|P(x) - f(x)| < \epsilon$$

und

$$P(x_i) = y_i$$

endlich genügt i_2 die folgenden Bedingungen

$$2i_1 < i_2, \log i_2 > 2^2\rho(2n_{2i_1})\cdot 2$$

Teilen wir nun das Intervall $-1 \leqq x \leqq +1$ in i_2 gleiche Teile. Nach Hilfssatz 1. können die gemeinsamen Nullstellen von je zwei der Tschebyscheffschen Polynome der Ordnung n_{i_2}, $2n_{i_2}$, n_{i_2+1}, $2n_{i_2+1}$, $\cdots$, n_{i_2+s}, $2n_{i_2+s}$, $\cdots$, n_{2i_2}, $2n_{2i_2}$ nur $\pm 1/\sqrt{2}$ und 0 sein. Also kann man wieder ein Polynom $\varphi_2(x)$ vom Grade n_{i_2} bestimmen das in den Nullstellen von ungeradem Index der Tschebyscheffschen Polynome n_{i_2+s}, $2n_{i_2+s}$-ter Ordnung (ausgenommen die etwaigen Nullstellen 0 und $\pm 1/\sqrt{2}$) wenn diese vom Endpunkte des s-ten Teilintervalls links liegen ($s = 1, 2, \cdots, i_2$), den Wert $+1$ annimmt; in den übrigen Nullstellen der Tschebyscheffschen Polynome n_{i_2+s}, $2n_{i_2+s}$-ter Ordnung und in den Nullstellen 0 und $\pm 1/\sqrt{2}$, gleich Null ist und im Intervall $-1 \leqq x \leqq +1$ $|\varphi_2(x)| \leqq 2$; endlich erfülle i_3 die folgenden Bedingungen:

$$2i_2 < i_3, \log i_3 > 2^3\rho(2n_{2i_2})\cdot 3$$

Ebenso definieren wir $\varphi_3(x)$, $\varphi_4(x)$, $\cdots$, $\varphi_r(x)$, $\cdots$. Allgemein gilt:

$$2i_{r-1} < i_r, \log i_r > 2^r\tau_r r, \quad |\varphi_r(x)| \leqq 2 \tag{10}$$

wo

$$\tau_r = \rho(2n_{2i_{r-1}}), \qquad \tau_1 = 1. \tag{11}$$

Bilden wir nun die Funktion

$$\varphi(x) = \sum_{r=1}^{\infty} \frac{\varphi_r(x)}{2^r\,\tau_r}$$

Es ist klar, dass $\varphi(x)$ im ganzen Intervall $-1 \leqq x \leqq +1$ stetig ist, und $|\varphi(x)| \leqq 2$. Wir beweisen, dass die zu den Tschebyscheffschen Abszissen gehörigen Lagrangeschen Interpolationspolynome der stetigen Funktion $\varphi(x)$ in sämtlichen Punkten des Intervalls $-1 < x \leqq +1$ divergieren, ja sogar unbeschränkt werden.

Es sei x ein beliebiger aber fixierter Punkt des Intervalls $-1 < x \leqq +1$. Zerlegen wir das Intervall $-1 < x \leqq +1$ in i_r gleiche Teile. Es liege x im $(s+1)$-ten Teilintervall. Bilden wir die Interpolationspolynome n_{i_r+s} und $2n_{i_r+s}$-ter Ordnung der Funktion $\varphi(x)$

$$\begin{aligned} L_{ni_r+s}[\varphi(x)] &= \sum_{k=1}^{\infty} L_{ni_r+s}\left[\frac{\varphi_k(x)}{2^k\,\tau_k}\right] \\ &= \sum_{k=1}^{r-1} L_{ni_r+s}\left[\frac{\varphi_k(x)}{2^k\,\tau_k}\right] + L_{ni_r+s}\left[\frac{\varphi_r(x)}{2^r\,\tau_r}\right] + \sum_{k=r+1}^{\infty} L_{ni_r+s}\left[\frac{\varphi_k(x)}{2^k\,\tau_k}\right] \\ &= \sum_1 + \sum_2 + \sum_3. \end{aligned} \tag{12}$$

$$L_{2n_{i_r+s}}[\varphi(x)] = \sum_{k=1}^{\infty} L_{2n_{i_r+s}}\left[\frac{\varphi_k(x)}{2^k \tau_k}\right]$$

$$= \sum_{k=1}^{r-1} L_{2n_{i_r+s}}\left[\frac{\varphi_k(x)}{2^k \tau_k}\right] + L_{2n_{i_r+s}}\left[\frac{\varphi_r(x)}{2^r \tau_r}\right] + \sum_{k=r+1}^{\infty} L_{2n_{i_r+s}}\left[\frac{\varphi_k(x)}{2^k \tau_k}\right] \tag{13}$$

$$= \sum_1' + \sum_2' + \sum_3'.$$

Nach Konstruktion sind aber für $k = 1, 2, \cdots, r-1$ die Polynome $\varphi_k(x)$ von höchstens n_{i_r}-tem Grade; daher ist

$$L_{n_{i_r+s}}[\varphi_k(x)] = \varphi_k(x), \qquad (k = 1, 2, \cdots, r-1)$$

$$L_{2n_{i_r+s}}[\varphi_k(x)] = \varphi_k(x), \qquad (k = 1, 2, \cdots, r-1)$$

und also

$$0 \leqq |\sum_1| = \left|\sum_{k=1}^{r-1} L_{n_{i_r+s}}\left[\frac{\varphi_k(x)}{2^k \tau_k}\right]\right| = \left|\sum_{k=1}^{r-1} \frac{\varphi_k(x)}{2^k \tau_k}\right| \leqq 2, \tag{14}$$

$$0 \leqq |\sum_1'| = \left|\sum_{k=1}^{r-1} L_{2n_{i_r+s}}\left[\frac{\varphi_k(x)}{2^k \tau_k}\right]\right| = \left|\sum_{k=1}^{r-1} \frac{\varphi_k(x)}{2^k \tau_k}\right| \leqq 2. \tag{15}$$

Nach der Definition von $\rho(n)$,

$$|\sum_3| = \left|\sum_{k=r+1}^{\infty} L_{n_{i_r+s}}\left[\frac{\varphi_k(x)}{2^k \tau_k}\right]\right| \leqq \sum_{k=r+1}^{\infty} \left|L_{n_{i_r+s}}\left[\frac{\varphi_k(x)}{2^k \rho(2n_{2ik})}\right]\right|$$

$$\leqq 2 \sum_{k=r+1}^{\infty} \frac{\varphi(n_{i_r+s})}{2^k \rho(2n_{2ik})} \leqq 2 \sum_{k=r+1}^{\infty} \frac{1}{2^k} < 1 \tag{16}$$

und ebenso

$$|\sum_3'| < 1. \tag{17}$$

Nun untersuchen wir noch Σ_2 und Σ_2'.

Das Interpolationspolynom n_{i_r+s}-ter Ordnung der Funktion $\varphi_r(x)$ stimmt, abgesehen von höchstens drei Gliedern (denen die etwaigen Nullstellen 0 oder $\pm 1/\sqrt{2}$ entsprechen) mit der im §1. abgeschätzten Summe $\lambda_{n_{i_r}}(x; i_r)$ ($m = i_r$, $n = n_{i_r}$) überein. Nun ist nach einem Satz von FEJÉR[5] im Falle Tschebyscheffscher Abszissen für jedes k und n

$$|l_k^{(n)}(x)| < \sqrt{2}$$

[5] LEOPOLD FEJÉR, „Lagrangesche Interpolation und die zugehörigen konjugierten Punkte," *Math. Annalen, 106* (1932), S. 1–55, insb. Formel (28).

woraus folgt, dass

$$\left|\sum\nolimits_2\right| \geqq \frac{1}{2^r \tau_r}\frac{1}{2\pi}\,|\cos(n_{i_{r+s}} \arccos x)|\log i_r - c_3(x) - 3\sqrt{2}$$

Ebenso

$$\left|\sum\nolimits_2'\right| \geqq \frac{1}{2^r \tau_r}\frac{1}{2\pi}\,|\cos(2n_{i_{r+s}} \arccos x)|\log i_r - c_3(x) - 3\sqrt{2}$$

also wegen (10)

$$\left|\sum\nolimits_2\right| \geqq \frac{r}{2\pi}\,|\cos(n_{i_{r+s}} \arccos x)| - c_3(x) - 3\sqrt{2}$$

$$= \frac{r}{2\pi}\,|\cos(n_{i_{r+s}} \arccos x)| - c_4(x),$$

$$\left|\sum\nolimits_2'\right| \geqq \frac{r}{2\pi}\,|\cos(2n_{i_{r+s}} \arccos x)| - c_4(x).$$

Endlich alle diese Ungleichungen zusammenfassend, ist

$$|L_{n_{i_{r+s}}}[\varphi(x)]| = |\Sigma_1 + \Sigma_2 + \Sigma_3| \geqq |\Sigma_2| - |\Sigma_1| - |\Sigma_3|$$

(18)
$$\geqq \frac{r}{2\pi}\,|\cos(n_{i_{r+s}} \arccos x)| - c_4(x) - 2 - 1,$$

$$|L_{2n_{i_{r+s}}}[\varphi(x)]| = |\Sigma_1' + \Sigma_2' + \Sigma_3'| \geqq |\Sigma_2'| - |\Sigma_1'| - |\Sigma_3'|$$

(19)
$$\geqq \frac{r}{2\pi}\,|\cos(2n_{i_{r+s}} \arccos x)| - c_4(x) - 2 - 1.$$

Nehmen wir nun an, dass die Lagrangeschen Interpolationspolynome der Funktion $\varphi(x)$ im Punkte x beschränkt blieben. In diesem Falle müsste wegen (18) und (19)

$$\lim_{r\to\infty} \cos(n_{i_{r+s}} \arccos x) = 0$$

$$\lim_{r\to\infty} \cos(2n_{i_{r+s}} \arccos x) = \lim_{r\to\infty} [2\cos^2(n_{i_{r+s}} \arccos x) - 1] = 0$$

was ersichtlich unmöglich ist.

4. Konstruktion einer stetigen Funktion, deren Interpolationspolynome in einem einzigen Punkt unbeschränkt werden.

Wir beweisen, dass es eine stetige Funktion $\psi(x)$ gibt, deren zu den Tschebyscheffschen Abszissen gehörigen Lagrangeschen Interpolationspolynome in sämtlichen Punkten des Intervalls $-1 < x \leqq +1$ beschränkt sind, aber im Punkte -1 unbeschränkt werden.

Bedeute $\rho(n)$ dasselbe wie im vorigen § und es sei $n_1 < n_2 < \cdots$ eine Folge der natürlichen Zahlen mit $(n_i, n_r) = 1 (i, r = 1, 2, \cdots)$.

Wir definieren die Funktion $\psi_{n_i}(x)$ folgendermassen. Sie sei gleich $+1$ in allen Nullstellen von ungeradem Index des Tschebyscheffschen Polynoms n_i-ter Ordnung auf welchem

$$1 + x_{2k+1}^{(n_i)} < 2/\sqrt{n_i}$$

gilt; in den übrigen Nullstellen des Tschebyscheffschen Polynoms n_i-ter Ordnung sei sie gleich 0.

Es sei endlich $\psi_{n_i}(x) \equiv 0$ im Intervall $-1 + 2/\sqrt{n_i} \leqq x \leqq +1$ und $\psi_{n_i}(x)$ etwa differenzierbar. Dann mit der Hilfe Hilfssatz 2 (mit der Methode des vorigen §) ist leicht beweisbar, dass die Interpolationspolynome der stetigen Funktion

$$\psi(x) = \sum_{i=1}^{\infty} \frac{\psi_{n_i}(x)}{i^2 \rho(n_{i-1})}$$

im Punkte -1 unbeschränkt werden, aber in den anderen Punkten des Intervalls $-1 < x \leqq +1$ beschränkt bleiben, wenn wir nur die Folge n_i geeignet wählen.

Endlich erwähnen wir noch, dass unser Verfahren auch im Falle anderer bekannter Grundpunkte erfolgreich ist; darauf kehren wir gelegentlich ein andersmal zurück.

BUDAPEST (HUNGARY)

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ON NON-HARMONIC FOURIER SERIES

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1. Wiener and Paley[1] have generalized many properties of the Fourier functions $\{e^{inx}\}$ to sets $\{e^{i\lambda_n x}\}$ where

$$|\lambda_n - n| \leqq D < \frac{1}{\pi^2}, \qquad -\infty < n < \infty, \tag{1.0}$$

for some fixed D.

They have proved that *if* (1.0) *is satisfied then the set* $\{e^{i\lambda_n x}\}$ *is closed in* $L^2(-\pi, \pi)$ *and possesses a unique biorthogonal set* $\{h_n(x)\}$ *such that for*

$$f(x) \epsilon L^2(-\pi, \pi)$$

the series

$$\sum_{-\infty}^{\infty} \left\{ \frac{e^{inx}}{2\pi} \int_{-\pi}^{\pi} f(\xi)\, e^{-in\xi}\, d\xi - e^{i\lambda_n x} \int_{-\pi}^{\pi} f(\xi)\, h_n(\xi)\, d\xi \right\} \tag{1.1}$$

converges uniformly to zero over any interval $(-\pi + \delta \leqq x \leqq \pi - \delta)$ *for any positive* δ, *and over any such interval the summability properties of*

$$\sum_{-\infty}^{\infty} e^{i\lambda_n x} \int_{-\pi}^{\pi} f(\xi)\, h_n(\xi)\, d\xi \tag{1.2}$$

are uniformly the same as those of the Fourier series of $f(x)$.

Wiener and Paley point out that whether or not $1/\pi^2$ in (1.0) is the best possible constant is not evident. Malin[2] has in fact shown that the theorem holds with $10/9\pi^2$ in place of $1/\pi^2$.

Wiener and Paley also ask whether or not the theorem can be generalized to the case where $f(x) \epsilon L$ rather than L^2.

It is the purpose of this paper to show that the theorem remains true if $1/\pi^2$ is replaced by $\frac{1}{4}$. Moreover it will be shown that this is the best possible constant. It will also be shown that there exists no general L theory but that there does exist an L^p theory where $1 < p \leqq 2$. These results are contained in the following theorems:

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[1] Wiener and Paley, "Fourier Transforms in the Complex Domain." Amer. Math. Soc. Coll. Pub. 19 (1934), Theorem XXXVIII.

[2] H. Malin. Doctor's dissertation at Mass. Inst. of Tech., 1934.

THEOREM I. *If* $\{\lambda_n\}$ *be a sequence of numbers and* D *a constant such that*

$$|\lambda_n - n| \leqq D < \frac{p-1}{2p}, \qquad -\infty < n < \infty, \tag{1.3}$$

for some fixed p, $1 < p \leqq 2$, *then the set* $\{e^{i\lambda_n x}\}$ *is closed* $L^p(-\pi, \pi)$ *and possesses a unique biorthogonal set* $\{h_n(x)\}$ *such that for any* $f(x) \epsilon L^p(-\pi, \pi)$ *the series* (1.1) *converges uniformly to zero over any interval* $(-\pi + \delta \leqq x \leqq \pi - \delta)$ *for any* $\delta > 0$. *Moreover the difference of the weighted sums (Riesz, Abel, etc.) of the non-harmonic and ordinary Fourier series also converges uniformly to zero over* $(-\pi + \delta \leqq x \leqq \pi - \delta)$.

In other words the convergence and summability properties of the series (1.2) are exactly the same as for the ordinary Fourier series for $f(x)$ over $-\pi < x < \pi$.

THEOREM II. *If* (1.3) *is replaced by*

$$|\lambda_n - n| \leqq \frac{p-1}{2p} \tag{1.4}$$

then the results of Theorem I no longer hold.

Combining these two theorems, it is clear that there exists no general L theory of non-harmonic Fourier series since as $p \to 1$, $D \to 0$.

In proving Theorem I the following result is of basic importance.

THEOREM III. *Let* $\{\lambda_n\}$ *satisfy* (1.3) *and let*

$$F(x) = \prod_1^\infty \left(1 - \frac{x}{\lambda_n}\right)\left(1 - \frac{x}{\lambda_{-n}}\right). \tag{1.5}$$

Then there exists an absolute constant A *such that*

$$\int_{-\infty}^{\infty} |F(x)|^p \, dx < \frac{A}{(p - 1 - 2pD)^2} \tag{1.6}$$

and

$$\int_c^{\infty} |F(x)|^p \, dx < \frac{Ac^{-(p-1-2pD)}}{(p - 1 - 2pD)^2} \tag{1.7}$$

for $c > 1$.

As we shall see, to prove that Theorem III is a best possible one is quite simple.

2. First we shall prove Theorem III. It is convenient to use $0 < A_k$, $k = 0, 1, \cdots$, for absolute constants. We now prove

LEMMA 1. *If* $F(x)$ *is defined as in Theorem* III *then*

$$\int_N^{2N} |F(x)|^p \, dx \leqq A_0 N^{-p+4pD} \int_N^{2N} \frac{|P(x)|^{2pD}}{|2N + 1 - x|^{2pD}} \, dx \tag{2.0}$$

where

$$P(x) = \prod_N^{2N} \left(\frac{\alpha_n + n - D - x - 2i}{n - D - x - 2i}\right) \tag{2.1}$$

and α_n *are constants such that* $0 \leqq \alpha_n \leqq 1$.

Let $N > 0$ be some integer. If $N < x < 2N$ it is clear from (1.3) that

$$1 + \frac{x}{n-D} \geqq \left|1 - \frac{x}{\lambda_{-n}}\right|, \qquad 0 < n < \infty,$$

$$\frac{x}{n-D} - 1 \geqq \left|\frac{x}{\lambda_n} - 1\right|, \qquad 0 < n < N, \tag{2.2}$$

$$1 - \frac{x}{n+D} \geqq \left|1 - \frac{x}{\lambda_n}\right|, \qquad 2N < n < \infty.$$

Therefore for $N < x < 2N$

$$|F(x)| \leqq \prod_1^{2N} \left|1 - \frac{x}{n-D}\right| \left|1 + \frac{x}{n-D}\right| \prod_{2N+1}^{\infty} \left|1 - \frac{x}{n+D}\right| \left|1 + \frac{x}{n-D}\right| \prod_N^{2N} \left|\frac{1 - \frac{x}{\lambda_n}}{1 - \frac{x}{n-D}}\right|.$$

And since for $n > 0$, $n - D \leqq \lambda_n$, it follows from this that

$$|F(x)| \leqq \prod_1^{2N} \left|1 - \frac{x}{n-D}\right| \left|1 + \frac{x}{n-D}\right| \prod_{2N+1}^{\infty} \left|1 - \frac{x}{n+D}\right| \left|1 + \frac{x}{n-D}\right| \prod_N^{2N} \left|\frac{\lambda_n - x}{n-D-x}\right|. \tag{2.3}$$

Let us recall the following well-known properties of the gamma function:

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

$$\Gamma(x+1) \sim (2\pi)^{\frac{1}{2}} x^{x+\frac{1}{2}} e^{-x}. \tag{2.4}$$

Using these on the right side of (2.3) we have for $N < x < 2N$

$$|F(x)| \leqq \frac{\Gamma^2(1-D)}{\pi} \frac{\Gamma(x+D)|\sin \pi(x+D)|\,\Gamma(2N+1+D)\Gamma(2N+1-D-x)}{\Gamma(x-D+1)\,\Gamma(2N+1-D)\,\Gamma(2N+1+D-x)} \times \prod_N^{2N} \left|\frac{\lambda_n - x}{n-D-x}\right|.$$

Using (2.4) we have

$$|F(x)| \leqq A_1 N^{-1+4D} \frac{|\sin \pi(x+D)|}{(2N+1-x)^{2D}} \prod_N^{2N} \left|\frac{\lambda_n - x}{n-D-x}\right|. \tag{2.5}$$

When $|n - D - x| < \frac{1}{2}$ we have

$$\left|\frac{\lambda_n - x}{n - D - x}\right| |\sin \pi(x + D)| \leqq \pi. \tag{2.6}$$

When $\frac{1}{2} \leqq |n - D - x|$ the inequality

$$1 + t \leqq e^t, \qquad -\infty < t < \infty, \tag{2.7}$$

gives

$$\begin{aligned}\left|\frac{\lambda_n - x}{n - D - x}\right| &= 1 + \frac{\lambda_n - n + D}{n - D - x} \leqq \exp \frac{\lambda_n - n + D}{n - D - x} \\ &< \left|\exp\left(\frac{\lambda_n - n + D}{n - D - x - 2i} + \frac{4D}{|n - D - x|^2}\right)\right|.\end{aligned} \tag{2.8}$$

Combining (2.6) and (2.8) we have

$$\begin{aligned}&\prod_N^{2N} \left|\frac{\lambda_n - x}{n - D - x}\right| |\sin \pi(x + D)| \\ &\qquad \leqq \pi \left|\exp\left(\sum_N^{2N} \frac{\lambda_n - n + D}{n - D - x - 2i} + 8D \sum_1^{\infty} \frac{1}{(n - \frac{1}{2})^2} + 2D\right)\right| \\ &\qquad < A_2 \left|\exp \sum_N^{2N} \frac{\lambda_n - n + D}{n - D - x - 2i}\right|.\end{aligned} \tag{2.9}$$

If we set $\alpha_n = (\lambda_n - n + D)/2D$ then clearly $0 \leqq \alpha_n \leqq 1$ and (2.9) becomes

$$\prod_N^{2N} \left|\frac{\lambda_n - x}{n - D - x}\right| |\sin \pi(x + D)| < A_2 \left|e^{\sum_N^{2N} \frac{\alpha_n}{n - D - x - 2i}}\right|^{2D}. \tag{2.10}$$

For $|z| \leqq \frac{1}{2}$ we have

$$|e^z| \leqq |1 + z|(1 + 2|z|^2). \tag{2.11}$$

Using this in (2.10) we get

$$\prod_N^{2N} \left|\frac{\lambda_n - x}{n - D - x}\right| |\sin \pi(x + D)| < A_3 \prod_N^{2N} \left|\frac{\alpha_n + n - D - x - 2i}{n - D - x - 2i}\right|^{2D}$$

If we use this in (2.5) we have (2.0).

LEMMA 2. Let

$$f(x) = \frac{(x - a_2 + ib) \cdots (x - a_{2k} + ib)}{(x - a_1 + ib) \cdots (x - a_{2k+1} + ib)} \tag{2.12}$$

where $0 \leqq a_1 < a_2 < \cdots < a_{2k+1} \leqq 1$ *and* $b \geqq 0$. *If* $0 \leqq t < 1$ *we have*[3]

$$\int_0^1 |f(x)|^t dx \leqq \frac{5}{1-t}. \tag{2.13}$$

Let

$$H(z) = i\,\frac{(z-a_2)\cdots(z-a_{2k})}{(z-a_1)\cdots(z-a_{2k+1})}.$$

Using the partial-fraction expansion for $H(z)$ we have

$$H(z) = i\left(\frac{c_1}{z-a_1} + \frac{c_3}{z-a_3} + \cdots + \frac{c_{2k+1}}{z-a_{2k+1}}\right)$$

where clearly $c_1 > 0, c_3 > 0, \cdots, c_{2k+1} > 0$. If we now set $z = x + iy$ we have

$$\Re H(z) = \frac{c_1 y}{(x-a_1)^2 + y^2} + \cdots + \frac{c_{2k+1} y}{(x-a_{2k+1})^2 + y^2}.$$

Thus $\Re H(z) \geqq 0$ for $y \geqq 0$, and therefore

$$|\operatorname{am} H(z)| \leqq \tfrac{1}{2}\pi, \qquad y \geqq 0. \tag{2.14}$$

Using (2.14) and

$$\Re H^t(z) = |H(z)|^t \cos(t \operatorname{am} H(z))$$

we get

$$|H(z)|^t \leqq \frac{\Re H^t(z)}{\cos \frac{1}{2}\pi t} < \frac{\Re H^t(z)}{1-t}. \tag{2.15}$$

Since $H(z)$ has no zeros or poles in $y > 0$, $H^t(z)$ is analytic for $y > 0$ and its integral around any closed contour in that region is zero. Thus for any fixed quantity c, $1 > c > 0$,

$$\int_{-1}^{2} dx\, H^t(x+ic) + i\int_c^1 dy\, H^t(2+iy)$$
$$-\int_{-1}^{2} dx\, H^t(x+i) - i\int_c^1 dy\, H^t(-1+iy) = 0.$$

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[3] The constant 5 is of no particular significance in this inequality. By using a more refined contour in the proof of this lemma we can show

$$\int_0^1 |f(x)|^t dx \leqq \frac{\pi^t}{\sin \frac{1}{2}\pi(1-t)}\left[\left(\frac{t}{1-t}\right)^{1-t} + \left(\frac{1-t}{t}\right)^t\right]. \tag{*}$$

As $t \to 1$ the right side of (*) approaches $2/(1-t)$. Since

$$\int_0^1 \frac{dx}{|x - \frac{1}{2}|^t} = \frac{2^t}{1-t}$$

the inequality (*) is asymptotically perfect as $t \to 1$.

Or

$$\left| \int_{-1}^{2} dx\, H^t(x+ic) \right| \leqq \int_0^1 dy \mid H(2+iy) \mid^t + \int_0^1 dy \mid H(-1+iy) \mid^t + \int_{-1}^{2} dx \mid H(x+i) \mid^t. \tag{2.16}$$

It is clear that for any fixed z

$$\mid H(z) \mid \leqq \frac{1}{\underset{0 \leqq l \leqq k}{\text{Min}} \mid z - a_{2l+1} \mid}. \tag{2.17}$$

Using this in (2.16) we have

$$\left| \int_{-1}^{2} dx\, H^t(x+ic) \right| \leqq 1 + 1 + 3 = 5.$$

Thus

$$\int_{-1}^{2} dx\, \Re H^t(x+ic) \leqq 5.$$

And using (2.15) we have

$$\int_{-1}^{2} dx \mid H(x+ic) \mid^t \leqq \frac{5}{1-t} \tag{2.18}$$

for any c, $1 > c > 0$. If $c \geqq 1$, (2.18) is trivial by (2.17). If $c = 0$, (2.18) holds since it holds for all $c > 0$. Thus (2.18) is true for all $c \geqq 0$ and (2.13) follows immediately.

Proof of Theorem III. If $P(x)$ is defined as in Lemma 1 and if we set $y = (x - N + 1)/(N + 2)$ we have

$$\int_{N-1}^{2N+1} \frac{\mid P(x) \mid^{2pD}}{\mid 2N + 1 - x \mid^{2pD}}\, dx$$

$$\leqq 4\, N^{1-2pD} \int_0^1 \frac{\mid y - a_2 + ib \mid^{2pD} \cdots \mid y - a_{2k} + ib \mid^{2pD}}{\mid y - a_1 + ib \mid^{2pD} \cdots \mid y - a_{2k+1} + ib \mid^{2pD}}\, dy$$

where $0 \leqq a_1 < a_2 \cdots < a_{2k+1} \leqq 1$ and where $b = 2/(N+2)$. If we now use (2.0) and Lemma 2 we have

$$\int_N^{2N} \mid F(x) \mid^p dx \leqq \frac{A_4\, N^{-(p-1-2pD)}}{1 - 2pD}.$$

If we set $N = 2^m, 2^{m+1}, \cdots$, and add we have

$$\int_{2^m}^{\infty} \mid F(x) \mid^p dx \leqq \frac{A_5\, 2^{-m(p-1-2pD)}}{(p - 1 - 2pD)(1 - 2pD)}.$$

The theorem now follows at once.

3. In proving Theorems I and II we require the following well-known theorem on Fourier transforms:

THEOREM A. *Let* $f(x) \epsilon L^p(-\infty, \infty)$ *for some fixed* p, $1 < p \leqq 2$. *Then*

$$g(u) = (2\pi)^{-\frac{1}{2}} \text{ l.i.m.}_{A\to\infty} \int_{-A}^{A} f(x)\, e^{iux}\, dx, \tag{3.0}$$

(*where* l.i.m. *is here of* $p/(p-1)$ *order*), *exists almost everywhere and*

$$f(x) = (2\pi)^{-\frac{1}{2}} \lim_{A\to\infty} \int_{-A}^{A} \left(1 - \frac{|u|}{A}\right) g(u)\, e^{-iux}\, du \tag{3.1}$$

almost everywhere. Moreover if $q = p/(p-1)$ *then*

$$\left[(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} |g(u)|^q\, du\right]^{1/q} \leqq \left[(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} |f(x)|^p\, dx\right]^{1/p}. \tag{3.2}$$

It is clear from (3.1) that

$$f(x) = (2\pi)^{-\frac{1}{2}} \lim_{A\to\infty} \int_{-A}^{A} g(u)\, e^{-iux}\, du \tag{3.3}$$

wherever this limit exists except for a set of measure zero.

As regards the closure of $\{e^{i\lambda_n x}\}$, it follows from results proved elsewhere[4] that the conditions of Theorem I imply the closure $L^p(-\pi, \pi)$ of $\{e^{i\lambda_n x}\}$. The closure of this set implies at once the uniqueness of $\{h_n(x)\}$ if it exists.

We now turn to the proofs of several lemmas concerning

$$G(w) = (w - \lambda_0) \prod_{1}^{\infty} \left(1 - \frac{w}{\lambda_n}\right)\left(1 - \frac{w}{\lambda_{-n}}\right) \tag{3.4}$$

where

$$|\lambda_n - n| \leqq D < \frac{p-1}{2p}. \tag{3.5}$$

We continue to use the A_k as absolute positive constants. In Lemma 3 we will not use all of (3.5) but merely $|\lambda_n - n| \leqq \frac{1}{4}$.

LEMMA 3. *If* $w = u + iv$ *then*

$$|G(w)| < A_{12}(|w| + 1)\, e^{\pi|v|}, \tag{3.6}$$

$$|G(w)| \geqq A_{13}|v|(1 + |w|)^{-\frac{1}{2}}\, e^{\pi|v|}, \tag{3.7}$$

and

$$|G(\tfrac{1}{2} + iv)| > A_{14}. \tag{3.8}$$

[4] Levinson, "On the Closure of $\{e^{i\lambda_n x}\}$," Theorem II. To appear in the Duke Journal of Mathematics.

If $u \geqq 0$, $|w| \geqq \frac{1}{2}$, and we choose an integer N determined by $N - \frac{1}{2} \leqq |w| < N + \frac{1}{2}$, then

$$\left|\frac{w}{\lambda_n}\right| < 1 \qquad \text{for } n > N.$$

Also, since $u \geqq 0$, w/λ_n lies in the right half plane. Therefore

$$\left|1 - \frac{w}{\lambda_n}\right| \geqq \left|1 - \frac{w}{n - \frac{1}{4}}\right|, \qquad n > N.$$

Similarly

$$\left|\frac{w}{\lambda_n}\right| > 1 \qquad \text{for } 0 < n < N$$

and therefore

$$\left|1 - \frac{w}{\lambda_n}\right| \geqq \left|1 - \frac{w}{n + \frac{1}{4}}\right|, \qquad 0 < n < N.$$

Obviously

$$\left|1 - \frac{w}{\lambda_{-n}}\right| \geqq \left|1 + \frac{w}{n + \frac{1}{4}}\right|, \qquad n > 0$$

and

$$\left|1 - \frac{w}{\lambda_N}\right| \geqq \frac{|v|}{N + \frac{1}{4}}.$$

Therefore

$$|G(w)| \geqq |u - \tfrac{1}{4} + iv| \frac{|v|}{N + \frac{1}{4}} \prod_1^{N-1} \left|1 - \frac{w}{n + \frac{1}{4}}\right| \left|1 + \frac{w}{n + \frac{1}{4}}\right|$$

$$\times \left|1 + \frac{w}{N + \frac{1}{4}}\right| \prod_{N+1}^{\infty} \left|1 - \frac{w}{n - \frac{1}{4}}\right| \left|1 + \frac{w}{n + \frac{1}{4}}\right|.$$

Using the gamma function in a manner similar to that in Lemma 1 we have

$$|G(w)| \geqq \frac{\Gamma^2(\frac{5}{4})}{\pi} |w - \tfrac{1}{4}| |v| \left|\frac{\sin \pi(w - \frac{1}{4})}{w - N - \frac{1}{4}}\right|$$

$$\times \left|\frac{\Gamma(w - \frac{1}{4})\,\Gamma(N + \frac{5}{4} - w)\,\Gamma(N + \frac{3}{4})}{\Gamma(w + \frac{5}{4})\,\Gamma(N + \frac{3}{4} - w)\,\Gamma(N + \frac{5}{4})}\right| \tag{3.9}$$

$$\geqq A_{15} \frac{|w|\,|v|\,e^{\pi|v|}}{1 + |w - N|} \left|\frac{\Gamma(w - \frac{1}{4})\,\Gamma(N + \frac{5}{4} - w)\,\Gamma(N + \frac{3}{4})}{\Gamma(w + \frac{5}{4})\,\Gamma(N + \frac{3}{4} - w)\,\Gamma(N + \frac{5}{4})}\right|.$$

But[5] for $x \geqq 0$

$$\log|\Gamma(x+iy)| = (x-\tfrac{1}{2})\log|x+iy| - y\tan^{-1}\frac{y}{x} - x + O(1) \tag{3.10}$$

uniformly as $|x+iy| \to \infty$. If we note that for $\alpha \geqq 0$ and $x \geqq 0$,

$$\left|y\tan^{-1}\frac{y}{x} - y\tan^{-1}\frac{y}{x+\alpha}\right| = \left|y\tan^{-1}\frac{\alpha y}{x(x+\alpha)+y^2}\right| < \left|y\tan^{-1}\frac{\alpha}{y}\right|$$

we have, using (3.10) in (3.9),

$$|G(w)| \geqq A_{16}\frac{|v|e^{\pi|v|}}{1+|w|^{\frac{1}{2}}}, \qquad |w| > \tfrac{1}{2},\ u \geqq 0.$$

Since $G(-w)$ is of the same form as $G(w)$, the results hold for all values of u and for $|w| > \frac{1}{2}$. By precisely the same argument with $|w| \leqq \frac{1}{2}$, and therefore with N now zero, it is obvious that the result is true for all w. Thus (3.7) is true. The other results of the lemma are carried through in a similar manner. (3.8) is of course a consequence of (3.7) when $|v|$ is not small.

LEMMA 4. *The functions* $\{g_n(x)\}$ *defined by*

$$g_n(x) = \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-A}^{A}\frac{G(u)}{(u-\lambda_n)G'(\lambda_n)}e^{-iux}\,du \tag{3.11}$$

form a biorthogonal sequence to $\{e^{i\lambda_n x}\}$, *and* $g_n(x) \in L^q$, $q = p/(p-1)$.

By Theorem III

$$\frac{G(u)}{u-\lambda_n} \in L^p.$$

Thus by Theorem A, $g_n(x)$ defined as in (3.11) exists.

For any fixed $x > \pi$ it follows at once from (3.6) that

$$\psi_n(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{G(u)e^{-iux}}{(u-\lambda_n)(u-i)}\,du = \lim_{A\to\infty}\frac{i}{2\pi}\int_C\frac{G(w)e^{-iwx}}{(w-\lambda_n)(w-i)}\,dw$$

where C is the closed contour formed by the line $-A \leqq u \leqq A$ and the semicircle $|w| = A$, $-\pi \leqq \operatorname{am} w \leqq 0$.

Since the integral of an analytic function around a closed contour is zero, it follows that $\psi_n(x)$ vanishes for $x > \pi$. But

$$g_n(x) = [G'(\lambda_n)]^{-1}\left(\frac{d}{dx}\psi_n(x) - \psi_n(x)\right)$$

---

[5] This is the real part of the complex form of Stirling's formula

$$\log\Gamma(z) = (z-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log 2\pi + O(1)$$

as $|z| \to \infty$ uniformly for $-\pi+\delta \leqq \operatorname{am} z \leqq \pi-\delta$, $\delta > 0$.

and therefore $g_n(x) = 0$ for $x > \pi$, and similarly for $x < -\pi$. Using Theorem A again we have, therefore,

$$\frac{G(u)}{(u-\lambda_n)G'(\lambda_n)} = \int_{-\pi}^{\pi} g_n(x)e^{iux}\,dx.$$

From this it follows at once that

$$\int_{-\pi}^{\pi} g_n(x)e^{i\lambda_m x}\,dx = \begin{cases} 0, & m \neq n \\ 1, & m = n. \end{cases}$$

PROOF OF THEOREM I. Let C denote a rectangular path in the complex ζ plane with vertices at $(N + \frac{1}{2} + iM, -N - \frac{1}{2} + iM, -N - \frac{1}{2} - iM, N + \frac{1}{2} - iM)$. Let $\phi(x) = 1$ for $|x| < 1$ and $\phi(x) = 0$ for $|x| > 1$. Let $G(u)$ be defined as in (3.4).

Using residues and recalling Lemma 4, we have

$$\begin{aligned} &\text{l.i.m.}_{A\to\infty} \frac{1}{4\pi^2 i}\int_{-A}^{A} du\, G(u)e^{-iuy}\int_C d\zeta\,\frac{e^{i\zeta x}}{G(\zeta)(u-\zeta)} \\ &= \text{l.i.m.}_{A\to\infty} \frac{1}{2\pi}\int_{-A}^{A} du\, G(u)e^{-iuy}\left[\sum_{-N}^{N}\frac{e^{i\lambda_n x}}{G'(\lambda_n)(u-\lambda_n)} - \frac{e^{iux}}{G(u)}\,\phi\left(\frac{u}{N+\frac{1}{2}}\right)\right] \qquad (3.12) \\ &= \sum_{-N}^{N} g_n(y)e^{i\lambda_n x} - \frac{\sin(N+\frac{1}{2})(x-y)}{\pi(x-y)}. \end{aligned}$$

If $f(x) \in L^p$ then

$$\begin{aligned} &\sum_{-N}^{N} e^{i\lambda_n x}\int_{-\pi}^{\pi} g_n(y)f(y)dy - \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\,\frac{\sin(N+\frac{1}{2})(x-y)}{x-y}\,dy \\ &= \frac{1}{4\pi^2 i}\int_{-\pi}^{\pi} dy\, f(y)\ \text{l.i.m.}_{A\to\infty}\int_{-A}^{A} du\, G(u)e^{-iuy}\int_C d\zeta\,\frac{e^{i\zeta x}}{G(\zeta)(u-\zeta)}. \end{aligned} \qquad (3.13)$$

We want to show that the right side of (3.13) tends to zero as N tends to infinity. Let

$$I_1(x) = \left|\int_{-\pi}^{\pi} dy f(y)\ \text{l.i.m.}_{A\to\infty}\int_{-A}^{A} du\, G(u)e^{-iuy}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} d\xi\,\frac{e^{i\xi x - Mx}}{G(\xi+iM)(u-\xi-iM)}\right|. \qquad (3.14)$$

This is in effect that part of the right side of (3.13) for which ζ varies over the upper horizontal side of the rectangle C. By Theorem III

$$\frac{G(u)}{u+i} \in L^p \qquad (3.15)$$

and therefore, using Hölder's inequality.

$$\frac{G(u)}{(u+i)^2} \in L. \qquad (3.16)$$

Since

$$\frac{1}{u-\xi-iM}=\frac{1}{u-iM}+\frac{\xi}{(u-iM)(u-\xi-iM)}$$

we have, using (3.16),

$$I_1(x)\leqq\left|\int_{-\pi}^{\pi}dy\,f(y)\,\text{l.i.m.}_{A\to\infty}\int_{-A}^{A}du\,\frac{G(u)e^{-iuy}}{u-iM}\right|\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}}d\xi\,\frac{e^{-Mx}}{|G(\xi+iM)|}$$

$$+\left|\int_{-\pi}^{\pi}dy\,f(y)\int_{-\infty}^{\infty}du\,\frac{G(u)e^{-iuy}}{u-iM}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}}d\xi\,\frac{\xi e^{i\xi x-Mx}}{G(\xi+iM)(u-\xi-iM)}\right|,$$

where the second term on the right is absolutely integrable. Since $f(y)\in L^p$ and since the transform of (3.15) converges in the q^{th} mean by Theorem A, we have if $g(u)$ is the transform of $f(y)$,

$$I_1(x)\leqq\left|\int_{-\infty}^{\infty}\frac{G(u)\,g(u)}{u-iM}\,du\right|\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}}d\xi\,\frac{e^{-Mx}}{|G(\xi+iM)|}$$

$$+\left|\int_{-\infty}^{\infty}du\,\frac{G(u)\,g(u)}{u-iM}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}}d\xi\,\frac{\xi\,e^{-Mx+i\xi x}}{G(\xi+iM)\,(u-\xi-iM)}\right|.$$

Or, for $M>1$,

$$I_1(x)\leqq 4\,N\,e^{M|x|}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}}\frac{d\xi}{|G(\xi+iM)|}\int_{-\infty}^{\infty}\left|\frac{G(u)\,g(u)}{u-\lambda_0}\right|du.$$

But by Theorem A, $g(u)\in L^q$. Using Hölder's inequality we get

$$I_1(x)\leqq 4\,BN\,e^{M|x|}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}}\frac{d\xi}{|G(\xi+iM)|}$$

where B is some constant depending on $f(x)$, p, and D. If we now use (3.7) of Lemma 3, we get

$$I_1(x)\leqq A_{17}\,BN\,(M^2+N^2)^2\,e^{-M(\pi-|x|)}. \tag{3.17}$$

Again let us consider that part of the right member of (3.13) for which ζ runs over the right vertical side of the rectangle C. We denote this by

$$I_2(x)=\left|\int_{-\pi}^{\pi}dy\,f(y)\,\text{l.i.m.}_{A\to\infty}\int_{-A}^{A}du\,G(u)\,e^{-iuy}\right.$$

$$\left.\int_{-M}^{M}d\eta\,\frac{e^{-x\eta}}{G(N+\frac{1}{2}+i\eta)\,(u-N-\frac{1}{2}-i\eta)}\right|.$$

If we split the range of the integral in u into the ranges $(-A, -2N)$, $(-2N, 2N)$, and $(2N, A)$, then, proceeding in a manner similar to that used with $I_1(x)$, we can invert the order of integration in y and u, where in this case we use

$$\frac{1}{u - N - \frac{1}{2} - i\eta} = \frac{1}{u - \lambda_0} + \frac{N + \frac{1}{2} + i\eta - \lambda_0}{(u - \lambda_0)(u - N - \frac{1}{2} - i\eta)},$$

on the ranges $(-A, -2N)$, $(2N, A)$. If we define $g(u)$ as before, we have

$$I_2(x) \leqq \left| \int_{-\infty}^{\infty} du\, G(u)\, g(u) \int_{-M}^{M} d\eta \frac{e^{-x\eta}}{G(N + \frac{1}{2} + i\eta)(u - N - \frac{1}{2} - i\eta)} \right|$$

$$\leqq \left| \int_{-\infty}^{\infty} du\, g(u + N + \tfrac{1}{2}) \int_{-M}^{M} d\eta \frac{G(u + N + \frac{1}{2})\, e^{-x\eta}}{G(i\eta + N + \frac{1}{2})(u - i\eta)} \right|.$$

Let

$$\mu_n = -N + \lambda_{N+n}$$

Then

$$|\mu_n - n| \leqq D$$

and

$$\frac{G(u + N + \frac{1}{2})}{G(i\eta + N + \frac{1}{2})} = \left(\frac{u + \frac{1}{2} - \mu_0}{i\eta + \frac{1}{2} - \mu_0}\right) \prod_1^{\infty} \left(\frac{u + \frac{1}{2} - \mu_n}{i\eta + \frac{1}{2} - \mu_n}\right)\left(\frac{u + \frac{1}{2} - \mu_{-n}}{i\eta + \frac{1}{2} - \mu_{-n}}\right).$$

Let

$$G_N(u) = (u - \mu_0) \prod_1^{\infty} \left(1 - \frac{u}{\mu_n}\right)\left(1 - \frac{u}{\mu_{-n}}\right).$$

Then

$$I_2(x) \leqq \left| \int_{-\infty}^{\infty} du\, g(u + N + \tfrac{1}{2})\, G_N(u + \tfrac{1}{2}) \int_{-M}^{M} d\eta \frac{e^{-x\eta}}{G_N(i\eta + \frac{1}{2})(u - i\eta)} \right|.$$

Clearly

$$\left|\frac{1}{u - i\eta}\right| < \frac{4}{|u + \frac{1}{2} - \mu_0|} \qquad \text{unless } |u| < 1 \text{ and } |\eta| < 1,$$

and therefore

$$I_2(x) \leqq 4 \int_{-\infty}^{\infty} du \left| \frac{g(u + N + \frac{1}{2})\, G_N(u + \frac{1}{2})}{u + \frac{1}{2} - \mu_0} \right| \int_{-\infty}^{\infty} d\eta \frac{e^{|x\eta|}}{|G_N(\frac{1}{2} + i\eta)|}$$

$$+ \int_{-1}^{1} du\, |g(u + N + \tfrac{1}{2})\, G_N(u + \tfrac{1}{2})| \left| \int_{-1}^{1} d\eta \frac{e^{-x\eta}}{(u - i\eta)\, G_N(\frac{1}{2} + i\eta)} \right|.$$

If we now make use of Lemma 3, we have for $|x| < \pi$

$$I_2(x) < \frac{A_{18}}{(\pi - |x|)^2} \int_{-\infty}^{\infty} du \left| \frac{g(u+N+\frac{1}{2})\, G_N(u+\frac{1}{2})}{u+\frac{1}{2}-\mu_0} \right| + \int_{-1}^{1} du\, |g(u+N+\tfrac{1}{2})\, G_N(u+\tfrac{1}{2})| \left| \int_{-1}^{1} d\eta \frac{e^{-x\eta}}{(u-i\eta)\, G_N(i\eta+\frac{1}{2})} \right|. \tag{3.18}$$

For $|\eta| \leqq 1$,

$$\left| \frac{e^{-x\eta}}{G_N(i\eta+\frac{1}{2})} - \frac{1}{G_N(\frac{1}{2})} \right| \leqq |\eta| \operatorname*{Max}_{|\eta|\leqq 1} \left| \frac{d}{d\eta} \frac{e^{-x\eta}}{G_N(i\eta+\frac{1}{2})} \right|.$$

Using (3.8) of Lemma 3, we have

$$\left| \frac{e^{-x\eta}}{G_N(i\eta+\frac{1}{2})} - \frac{1}{G_N(\frac{1}{2})} \right| \leqq |\eta| \left(\frac{\pi e^{\pi}}{A_{14}} + \frac{e^{\pi}}{A_{14}^2} \operatorname*{Max}_{|\eta|<1} |G_N'(i\eta+\tfrac{1}{2})| \right).$$

But

$$G_N'(i\eta+\tfrac{1}{2}) = \frac{1}{2\pi i} \int_C \frac{G_N(z)}{(z-i\eta-\frac{1}{2})^2}\, dz$$

where C can be taken as a circle of radius 10 about the origin. Using (3.6) of Lemma 3 we see that there exists an A_{19} such that $|G_N'(i\eta+\frac{1}{2})| < A_{19}$, $|\eta| < 1$. Therefore

$$\left| \frac{e^{-x\eta}}{G_N(\frac{1}{2}+i\eta)} - \frac{1}{G_N(\frac{1}{2})} \right| \leqq A_{20} |\eta|, \qquad |\eta| \leqq 1.$$

Using this we have

$$\left| \int_{-1}^{1} d\eta \frac{e^{-x\eta}}{(u-i\eta)\, G_N(i\eta+\frac{1}{2})} \right| \leqq \frac{1}{|G_N(\frac{1}{2})|} \left| \int_{-1}^{1} \frac{d\eta}{\eta+iu} \right| + A_{20} \int_{-1}^{1} \frac{|\eta|}{|\eta+iu|}\, d\eta$$

$$\leqq \frac{\pi}{A_{14}} + 2A_{20} < A_{21}.$$

Using this in (3.18) we have

$$I_2(x) \leqq \frac{A_{18}}{(\pi-|x|)^2} \int_{-\infty}^{\infty} \left| \frac{g(u+N+\frac{1}{2})\, G_N(u+\frac{1}{2})}{u+\frac{1}{2}-\mu_0} \right| du + A_{21} \int_{-1}^{1} |g(u+N+\tfrac{1}{2})|\, du.$$

We know from Theorem A that

$$\int_{-\infty}^{\infty} |g(u)|^q\, du < \infty. \tag{3.19}$$

Using Hölder's inequality, we get

$$I_2(x) \leqq \frac{A_{18}}{(\pi - |x|)^2}\left[\int_{-\infty}^{-\frac{1}{4}N}\left|\frac{G_N(u+\frac{1}{2})}{u+\frac{1}{2}-\mu_0}\right|^p du\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|g(u)|^q du\right]^{\frac{1}{q}}$$
$$+\frac{A_{18}}{(\pi - |x|)^2}\left[\int_{\frac{1}{4}N}^{\infty}|g(u+\tfrac{1}{2})|^q du\right]^{\frac{1}{q}}\left[\int_{-\infty}^{\infty}\left|\frac{G_N(u+\frac{1}{2})}{u+\frac{1}{2}-\mu_0}\right|^p du\right]^{\frac{1}{p}}$$
$$+2A_{21}\left[\int_{-1}^{1}|g(u+N+\tfrac{1}{2}|^q du\right]^{\frac{1}{q}}.$$

If we now use (3.19) and Theorem III, we get

$$\lim_{N\to\infty} I_2(x) = 0, \qquad |x| \leqq \pi - \delta, \qquad \delta > 0, \tag{3.20}$$

uniformly in x. This holds entirely independently of M.

Let us denote that part of the integral on the right side of (3.13) for which ζ varies along the lower horizontal side of the rectangle C by $I_3(x)$. Then clearly $I_3(x)$ satisfies the inequality (3.17) of $I_1(x)$. We also introduce an $I_4(x)$ similarly related to $I_2(x)$.

By (3.20), we can make $I_2(x)$, and therefore $I_2(x) + I_4(x)$, arbitrarily small independently of M by choosing N sufficiently large. Having done this we can, for the particular value of N, make $I_1(x) + I_3(x)$ arbitrarily small by making M sufficiently large. This proves that

$$\lim_{N\to\infty}\left[\sum_{-N}^{N} e^{i\lambda_n x}\int_{-\pi}^{\pi} g_n(y)\,f(y)\,dy - \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\,\frac{\sin(N+\frac{1}{2})(x-y)}{x-y}\,dy\right] = 0$$

uniformly in x for $|x| \leqq \pi - \delta, \delta > 0$.

To show that the summability properties of the two series are the same requires a slight variation of the ideas used above. We will take the case of Riesz summability of order α. How to proceed with various other types of summability will be obvious from this. We suppose that $\lambda_0 \neq 0$.

Let C_1 represent the rectangle with vertices at $-iM, N+\frac{1}{2}-iM, N+\frac{1}{2}+iM, i\,M$ and let C_2 be the reflection of C_1 in the imaginary axis. Then as in (3.12)

$$\frac{1}{4\pi^2 i}\,\text{l.i.m.}_{A\to\infty}\int_{-A}^{A} du G(u)e^{-iuy}\left[\int_{C_1}\frac{e^{i\zeta x}}{G(\zeta)(u-\zeta)}\left(1-\frac{\zeta}{N+\frac{1}{2}}\right)^{\alpha} d\zeta\right.$$
$$\left.+\int_{C_2}\frac{e^{i\zeta x}}{G(\zeta)(u-\zeta)}\left(1+\frac{\zeta}{N+\frac{1}{2}}\right)^{\alpha} d\zeta\right] \tag{3.21}$$
$$= \sum_{-N}^{N} g_n(y)e^{i\lambda_n x}\left(1-\frac{|\lambda_n|}{N+\frac{1}{2}}\right)^{\alpha} - \frac{1}{\pi}\int_0^1 \cos\frac{u(x-y)}{N+\frac{1}{2}}\,(1-u)^{\alpha}\,du.$$

If we now multiply through by $f(y)$ and integrate as in (3.13), we have to show that the repeated integral analogous to the left side of (3.13) tends uniformly to zero as $N \to \infty$ if $|x| \leqq \pi - \delta$. That this is true is clear if we observe that

the integral from $\zeta = iM$ to $\zeta = -iM$ in C_1 is cancelled by the integral from $\zeta = -iM$ to $\zeta = iM$ in C_2 in (3.21) and thus we are really only concerned with ζ as it varies over the sides of the rectangle C of (3.13). This we can handle in a manner similar to that used on (3.13). Thus

$$\lim_{N\to\infty} \sum_{-N}^{N} \left[e^{i\lambda_n x} \left(1 - \frac{|\lambda_n|}{N+\frac{1}{2}}\right)^{\alpha} \int_{-\pi}^{\pi} f(y) g_n(y)\, dy \right.$$

$$\left. - e^{inx} \left(1 - \frac{|n|}{N+\frac{1}{2}}\right)^{\alpha} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dy \right] = 0$$

uniformly for $|x| \leqq \pi - \delta$ if $\lambda_0 \neq 0$.

If $\lambda_0 = 0$ we can modify the contours C_1 and C_2 by taking a small semi-circular path of radius ρ around the origin. In this case we have to consider not only that part of (3.21) for which ζ varies over C but also a part involving

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \rho e^{i\theta} \frac{e^{i\zeta x}}{G(\zeta)(u-\zeta)} \left[\left(1 - \frac{\zeta}{N+\frac{1}{2}}\right)^{\alpha} - \left(1 + \frac{\zeta}{N+\frac{1}{2}}\right)^{\alpha} \right]$$

where $\zeta = \rho e^{i\theta}$.

The fact that

$$\left(1 - \frac{\zeta}{N+\frac{1}{2}}\right)^{\alpha} - \left(1 + \frac{\zeta}{N+\frac{1}{2}}\right)^{\alpha}$$

has a zero at $\zeta = 0$ is what allows us to discard this part of the left side of (3.21).

Thus the Riesz summability of the two series are the same. The same method applies to Abel and many other types of summability.

We have completed the proof of Theorem I, since we have shown the existence of a set $\{g_n(x)\}$ satisfying all the requirements of $\{h_n(x)\}$. Therefore we now take for $\{h_n(x)\}$ in Theorem I the set $\{g_n(x)\}$.

4. Proof of Theorem II. Here we show that the set

$$\lambda_n = n - \frac{p-1}{2p}, \qquad n > 0,$$

$$\lambda_0 = 0, \tag{4.0}$$

$$\lambda_{-n} = -n + \frac{p-1}{2p}, \qquad n > 0,$$

which obviously satisfies (1.4), does not have the properties which we have shown sets satisfying condition (1.3) to have.

Let us suppose that $\{e^{i\lambda_n x}\}$ has a biorthogonal set $\{h_n(x)\}$ and let us consider $h_0(x)$. Clearly $h_0(x)$ must be integrable. Therefore there exists

$$H(w) = \int_{-\pi}^{\pi} h_0(x) e^{iwx}\, dx \tag{4.1}$$

and moreover

$$H(\lambda_n) = 0, \qquad n \neq 0,$$

and $H(0) = 1$. Therefore

$$H(w) = \phi(w) \prod_1^\infty \left(1 - \frac{w^2}{\left(n - \frac{p-1}{2p}\right)^2}\right) \tag{4.2}$$

where $\phi(w)$ is an entire function. From (4.1),

$$| H(u + iv) | \leqq e^{\pi|v|} \int_{-\pi}^{\pi} | h_0(x) | \, dx < Ce^{\pi|v|} \tag{4.3}$$

for some constant C.

Using the gamma function we have

$$\begin{aligned} g(w) &= \prod_1^\infty \left(1 - \frac{w^2}{\left(n - \frac{p-1}{2p}\right)^2}\right) \\ &= \frac{\Gamma^2\left(\frac{p+1}{2p}\right)}{\pi} \frac{\Gamma\left(w + \frac{p-1}{2p}\right) \sin \pi\left(w + \frac{p-1}{2p}\right)}{\Gamma\left(w + \frac{p+1}{2p}\right)}. \end{aligned} \tag{4.4}$$

Using (3.10) we have for $| v | > 1$

$$| g(u + iv) | > A_{22} \frac{e^{\pi|v|}}{| u + iv |^{1/p} + 1}. \tag{4.5}$$

Using this and (4.3) in $H(w) = \phi(w)g(w)$, we have

$$CA_{22}^{-1}(| w |^{1/p} + 1) > | \phi(w) |, \qquad | v | > 1. \tag{4.6}$$

Using the Hadamard product theorem to write $H(w)$ as an infinite product, it is clear that $\phi(w)$ is at most of order 1. Thus by a well-known result of Pragmén and Lindelöf, (4.6) implies that $\phi(w) = O(| w |^{1/p})$. Since $p > 1$ it follows that $\phi(w)$ is a constant. Thus (4.1) becomes

$$\int_{-\pi}^{\pi} h_0(x) e^{iux} \, dx = g(u).$$

By (3.3) of Theorem A

$$h_0(x) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} g(u) e^{-iux} \, du$$

(wherever this limit is defined). Since $g(u)$ is even, we have

$$h_0(x) = \lim_{A \to \infty} \frac{1}{\pi} \int_0^A g(u) \cos ux \, du.$$

Or

$$h_0(x) = \frac{\Gamma^2\left(\frac{p+1}{2p}\right)}{\pi^2} \lim_{A\to\infty} \int_0^A \cos ux \frac{\Gamma\left(u+\frac{p-1}{2p}\right)\sin \pi\left(u+\frac{p-1}{2p}\right)}{\Gamma\left(u+\frac{p+1}{2p}\right)}\, du.$$

We now use a more refined form of Stirling's formula which for real u gives

$$\Gamma(u+1) = (2\pi)^{\frac{1}{2}} u^{u+\frac{1}{2}} e^{-u}\left[1 + O\left(\frac{1}{u}\right)\right]$$

as $u \to \infty$. Using this we have

$$h_0(x) = C_1 \lim_{A\to\infty} \int_0^A \cos ux \sin \pi\left(u+\frac{p-1}{2p}\right) u^{-1/p}\, du + O(1)$$

where

$$C_1 = e^{1/p}\Gamma^2\left(\frac{p+1}{2p}\right)\pi^{-2}.$$

But

$$2\cos ux \sin \pi\left(u+\frac{p-1}{2p}\right) = \sin\left[\pi\left(u+\frac{p-1}{2p}\right)+ux\right] + \sin\left[\pi\left(u+\frac{p-1}{2p}\right)-ux\right].$$

Thus for $x \geqq 0$

$$h_0(x) = \tfrac{1}{2}C_1 \lim_{A\to\infty} \int_0^A \sin\left[\pi\left(u+\frac{p-1}{2p}\right)-ux\right] u^{-1/p}\, du + O(1).$$

Therefore for $\pi > x \geqq 0$

$$h_0(x) = \tfrac{1}{2}C_1 \sin\frac{\pi}{p}\,\Gamma\left(1-\frac{1}{p}\right)(\pi - x)^{-1+1/p} + O(1).$$

Since $h_0(x)$ is even, we have

$$h_0(x) = C_2(\pi - |x|)^{-1+1/p} + O(1), \qquad |x| < \pi.$$

If

$$f(x) = \frac{1}{(\pi - |x|)^{1/p}\log(\pi - |x|)}$$

for $\frac{1}{2}\pi < |x| < \pi$ and zero elsewhere, then

$$\int_{-\pi}^{\pi} |f(x)|^p\, dx < \infty.$$

But clearly

$$\int_{-\pi}^{\pi} h_0(x)f(x)\,dx = \infty.$$

Therefore the conclusions of Theorem I cannot hold for the set $\{\lambda_n\}$ given by (4.0). In other words, condition (1.3) cannot be relaxed to (1.4) without disrupting the entire theorem.

Incidentally since for $u \to \infty$

$$g(u) \sim C_3 u^{-1/p} \sin \pi \left(u + \frac{p-1}{2p}\right)$$

it is obvious that

$$\int_1^{\infty} |\, g(u)\,|^p\, du = \infty$$

which shows that Theorem III is a best possible theorem.

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